SPACE AND TIME TRADEOFFS

The basic idea behind space and time tradeoffs is to preprocess the input of the problem and store the additional information obtained. This helps in solving the problem. This approach is known as ***input enhancement***. We will discuss following algorithms based on input enhancement:

- Counting methods for sorting
- Boyer-Moore Algorithm for string matching
- Horspool Algorithm for string matching

Another technique that uses space and time tradeoffs suggests using extra space to facilitate faster and/or more flexible access to the data. This approach is known as ***prestructuring.*** This indicates that, some processing is done before a problem in question is actually solved, but unlike input enhancement, it deals with access structuring. We will illustrate this approach by

- Hashing

Sorting by Counting

In this method is the application of input enhancement. Here, we will count the number of elements smaller than each element. This count is stored in a table and it will indicate the position of that element in the sorted list. This algorithm is known as ***comparison counting.***

```
ALGORITHM ComparisonCounting(A[0…n-1])
//Sorts an array by comparison counting
//Input: Array A[0…n-1]
//Output: Array S[0…n-1] of A's elements in a sorted order

for i ← 0 to n-1 do
        Count[i] ← 0

for i ← 0 to n-2 do
        for j ← i+1 to n-1 do
                if A[i] < A[j]
                        Count[j] ← Count[j] +1
                else
                        Count[i] ← Count[i] +1

for i ← 0 to n-1 do
        S[Count[i]] ← A[i]

return S
```

Analysis:

The basic operation is comparison. Thus, the time complexity can be given as –

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1$$

$$= \frac{n(n-1)}{2}$$

Thus,

$$C(n) \in \theta(n^2)$$

Input Enhancement in String Matching

String matching problem requires finding an occurrence of a given pattern of *m* characters in a given text of *n* characters. We have seen Brute force technique for solving this problem. Here we will study Boyer-Moore algorithm and its simplified version, Horspool algorithm for string matching.

Consider as an example, searching for the pattern BARBAR in some text:

s_0, ... c........................... s_{n-1}
BARBAR

Starting with last character *R* of the pattern, we have to keep on comparing each pair of characters in pattern and text. If all the characters match, then algorithm halts. If any mismatch is found, we need to shift our pattern towards right. The number of characters to be shifted depends on various situations as discussed here under:

Case 1. If there are no *c*'s in the pattern, shift pattern by its entire length. For example,

s_0, - - - - - - - - - - - - S - s_{n-1}
≠
BARBAR ↓
BARBAR

Here, S ≠ R and also S is not present in the pattern. So, shift entire pattern.

Case 2. If there are occurrences of character *c* in pattern, but it is not last character of the pattern, then shift should align the rightmost occurrence of *c* in the pattern with the *c* in text. For example,

s_0, - - - - - - - - - - - - B - s_{n-1}
≠
BARBAR
BARBAR

Case 3. If *c* is the last character in the pattern, but there are no *c*'s among other m-1 characters, then there will be entire pattern shift. For example,

s_0, - - - - - - - - - - M A R - s_{n-1}
≠ = = ↓
BAR B A R
BARBAR

Case 4. If *c* is the last character in pattern, and also there are some other *c*'s in the pattern, then shift will be same as in case2. For example,

s_0, - - - - - - - - - - P R - s_{n-1}
≠ =
BARBAR
BARBAR

In Horspool and Boyer-Moore algorithm, we have to pre-compute the shift sizes and store them in a table. The shift table will be indexed by all possible characters that can be encountered in text. The table entries will be filled with shift sizes.

Specifically, for every character *c* in text, we compute the shift's value by the formula –

$$t(c) = \begin{cases} \text{the pattern length m, if } c \text{ is not among first m-1 characters of pattern} \\ \text{the distance from the rightmost } c \text{ among first m-1 characters of the pattern to its last character, otherwise.} \end{cases}$$

Following the algorithm for calculating shift-table values.

```
ALGORITHM ShiftTable(P[0…m-1])
//Fills the shift table used by Horspool's and Boyer-Moore algorithms
//Input: Pattern P[0…n-1] and an alphabet of possible characters
//Output: Table[0…size-1] indexed by the alphabet's characters and filled with
// shift sizes.

Initialize all the elements of Table with m

for j ← 0 to m-2 do
        Table[P[j]] ← m-1- j

return Table
```

Now, the algorithm for Horspool technique can be summarized as below –

Step 1. For a given pattern of length *m* and the alphabet used in both the pattern and text, construct the shift table.
Step 2. Align the pattern against the beginning of the text.
Step 3. Repeat the following until either a matching substring is found or the pattern reaches beyond the last character of the text.

Starting with the last character in the pattern, compare the corresponding characters in the pattern and text until either all *m* characters are matched or a mismatching pair is encountered.

In the latter case, retrieve the entry *t(c)* from the *c*'s column of the shift table where *c* is the text's character currently aligned against the last character of the pattern, and shift the pattern by *t(c)* characters to the right along the text.

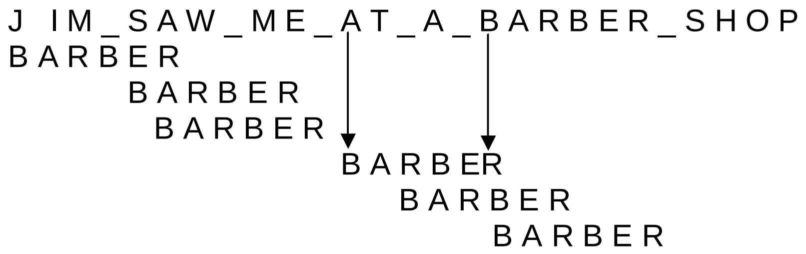
The pseudocode can be given as –

```
ALGORITHM Horspool(P[0…m-1], T[0…n-1])
//Implement Horspool's algorithm for string matching
//Input: Pattern P[0…m-1] and text T[0…n-1]
//Output: The position of first matching, if successful, otherwise, -1

ShiftTable(P[0…m-1])

i←m-1
while i ≤ n-1 do
    k ← 0
    while k ≤ m-1 and P[m-1-k] ==T[i-k]
        k ← k+1
    if k==m
        return i-m+1
    else
        i ← i+ Table[T[i]]

return -1
```

Example:
Let us consider an example of search a pattern BARBER in the text –
JIM_SAW_ME_AT_A_BARBER_SHOP

The text consists of the alphabets A to Z and a character _.
Let us construct a shift-table.

Character *c*	A	B	C	D	E	F	….	R	….	Z	---
Shift *t(c)*	4	2	6	6	1	6	6	3	6	6	6



HASHING

Hashing is a way of representing dictionaries. Dictionary is an abstract data type with a set of operations searching, insertion and deletion defined on its elements. The elements of dictionary can be numeric or characters or most of the times, records.

Usually, a record consists of several fields; each may by different data types. For example, student record may contain student id, name, gender, marks etc. Every record is usually identified by some ***key***. Here we will consider the implementation of a dictionary of *n* records with keys k1, k2 …kn.

Hashing is based on the idea of distributing keys among a one-dimensional array H[0…m-1], called ***hash table***. For each key, a value is computed using a predefined function called ***hash function***. This function assigns an integer, called ***hash address***, between 0 to m-1 to each key. Based on the hash address, the keys will be distributed in a hash table.

For example, if the keys k1, k2, …., kn are integers, then a hash function can be –

h(K) = K mod m.

Let us take keys as 65, 78, 22, 30, 47, 89. And let hash function be , h(k) = k %10.
Then the hash address may be any value from 0 to 9 and hash table may look like –

0	1	2	3	4	5	6	7	8	9

For each key, hash address will be computed as –

h(65) = 65 %10 = 5
h(78) = 78%10 = 8
h(22)= 22 % 10 =2
h(30)= 30 %10 =0
h(47) = 47 %10 = 7
h(89)=89 % 10 = 9

Now, each of these keys can be hashed into a hash table as –

0	1	2	3	4	5	6	7	8	9
30		22			65		47	78	89

In general, a hash function should satisfy the following requirements:

- A hash function needs to distribute keys among the cells of hash table as evenly as possible.
- A hash function has to be easy to compute.

Hash Collisions

Let us have *n* keys and the hash table is of size *m* such that *m<n.* As each key will have an address with any value between 0 to m-1, it is obvious that more than one key will have same hash address. That is, two or more keys need to be hashed into the same cell of hash table. This situation is called as ***hash collision***. In the worst case, all the keys may be hashed into same cell of hash table. But, we can avoid this by choosing size of hash table and hast function properly. Anyway, every hashing scheme must have a mechanism for resolving hash collision.

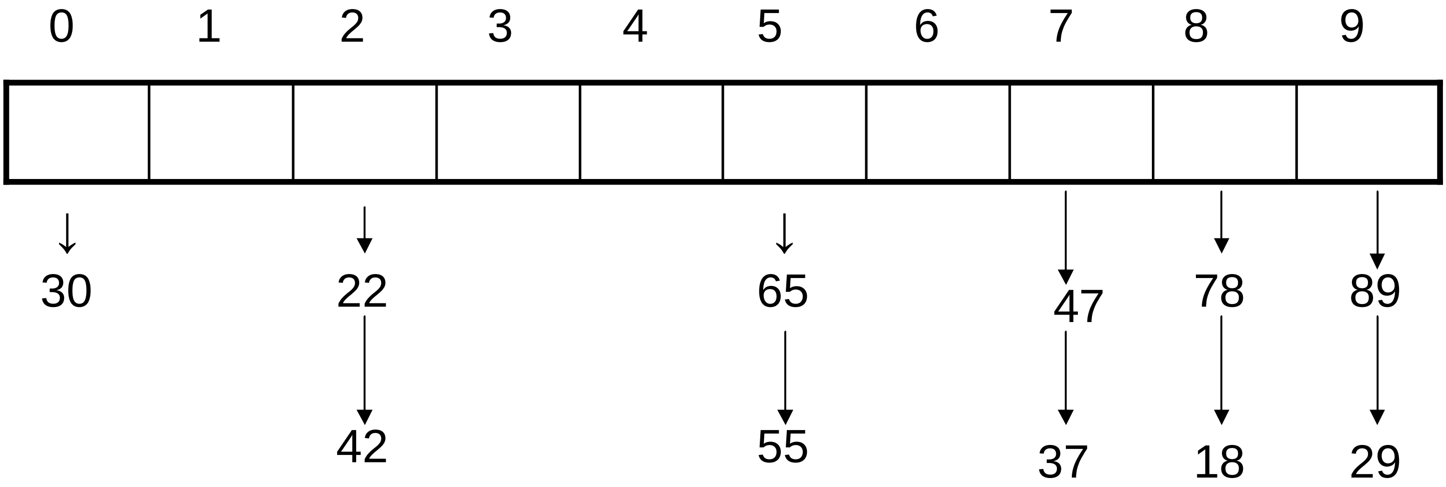
There are two methods for hash collision resolution, viz Open hashing and closed hashing.

Open Hashing (Separate Chaining)

- In open hashing, keys are stored in linked lists attached to cells of a hash table.
- Each list contains all the keys hashed to its cell.
- For example, consider the elements 65, 78, 22, 30, 47, 89, 55, 42, 18, 29, 37.
- If we take the hash function as h(k)= k%10, then the hash addresses will be –

  h(65) = 65 %10 = 5
  h(22)= 22 % 10 =2
  h(47) = 47 %10 = 7
  h(55)=55%10 =5
  h(78) = 78%10 = 8
  h(30)= 30 %10 =0
  h(89)=89 % 10 = 9
  h(42)=42%10 =2
  h(18)=18%10 =8
  h(29)=29%10=9
  h(37)=37%10 =7

- Now, the hashing is done as below –



Operations:

- **Searching:** Now, if we want to search for the key element in a hash table, we need to find the hash address of that key using same hash function.
- Using the obtained hash address, we need to search the linked list by tracing it, till either the key is found or list gets exhausted.
- **Insertion:** Insertion of new element to hash table is also done in similar manner.
- Hash key is obtained for new element and is inserted at the end of the list for that particular cell.
- **Deletion:** Deletion of element is done by searching that element and then deleting it from a linked list.

Efficiency:

- If the hash function distributes *n* keys among *m* cells of the hash table about evenly, then each linked list will be about n/m keys long.
- The ratio $\alpha = n/m$ is called as ***load factor***.
- The average number of comparisons done for a successful search, $S \approx 1 + \alpha/2$
- And for unsuccessful search, $U = \alpha$

Closed Hashing (Open Addressing)

In this technique, all keys are stored in the hash table itself without using linked lists. Different methods can be used to resolve hash collisions. The simplest technique is ***linear probing.*** This method suggests to check the next cell from where the collision occurs. If that cell is empty, the key is hashed there. Otherwise, we will continue checking for the empty cell in a circular manner. Thus, in this technique, the hash table size must be at least as large as the total number of keys.
Consider the elements 65, 78, 18, 22, 30, 89, 37, 55, 42

Let us take the hash function as h(k)= k%10, then the hash addresses will be –

h(65) = 65 %10 = 5 h(78) = 78%10 = 8 h(18)=18%10 =8
h(22)= 22 % 10 =2 h(30)= 30 %10 =0 h(89)=89 % 10 = 9
h(37)=37%10 =7 h(55)=55%10 =5 h(42)=42%10 =2

Now, hashing is done as below –

0	1	2	3	4	5	6	7	8	9
30	89	22	42		65	55	37	78	18

Efficiency:

- If the hash function distributes *n* keys among *m* cells of the hash table about evenly, then each linked list will be about n/m keys long.
- The ratio α =n/m is called as ***load factor***.
- The average number of comparisons done for a successful search,

$$S \approx \frac{1}{2}\left(1+\frac{1}{1-\alpha}\right)$$

- And for unsuccessful search,

$$U \approx \frac{1}{2}\left(1+\frac{1}{(1-\alpha)^2}\right)$$

unit-7
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DYNAMIC PROGRAMMING

We know that divide-and-conquer technique is used to solve the problems that can be divided into independent subproblems. On the other hand, dynamic programming is one such strategy that can be used to solve the problems having dependent subproblems. That is, in case of some problems, their subproblems are shared and they can not be solved independently. In case of some other problems, even though we can solve subproblems independently, many of the calculations may repeat, thus increasing the time.

Consider a problem of finding n^{th} fibonacci number. The formula is given by-

$$F(n) = F(n-1) + F(n-2)$$

with initial conditions, $F(0)=0$ & $F(1)=1$.

Here, if we try to solve $F(n-1)$, that will contain a term $F(n-2)+F(n-3)$. So, $F(n)$ and its subproblem $F(n-1)$ are sharing another subproblem $F(n-2)$. Thus, calculating these repeated terms is simply a waste of time.

So, instead of solving overlapping subproblems again and again, dynamic programming suggests solving each of the smaller subproblems only once and recording the results in a table from which we can obtain a solution for the original problem.

For example, to compute nth Fibonacci number, we can use the initial conditions $F(0)=0$ & $F(1)=1$ first and generate consecutive fibonacci numbers.

Computing Binomial Coefficient

It is a best example of dynamic programming strategy. We know that, binomial coefficient ${}^nC_k$ is the number of combinations of k elements from n elements. Binomial coefficients can be obtained from the formula-

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + \dots$$
$$\qquad {}^nC_i\, a^{n-i} b^i + \dots + {}^nC_n\, a^0 b^n$$

But, here, we will consider a recursive formula

$${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k \quad , n>k>0$$
$$\& \quad {}^nC_0 = {}^nC_n = 1$$

It is obvious that the above relation is a recurrence relation and ${}^{n}C_k$ is computed interms of the smaller overlapping problems of same type. So, dynamic programming strategy suggests to prepare a table with n+1 rows and k+1 columns as below and the entries of the table are made based on above equation.

	0	1	2	3 ..	k-1	. . . .	k
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
⋮	⋮						
k	1						1
⋮	⋮						
n-1	1				${}^{n-1}C_{k-1}$		${}^{n-1}C_k$
n	1						${}^{n}C_k$

Here, to get a particular cell value, we will add the entry at the same column and previous row and the entry at previous column and previous row.

NOTE: This triangular shape table is also known as Pascal's triangle.

The algorithm is as below-

```
ALGORITHM Binomial(n, k)
// To compute binomial coefficient $^nC_k$ by
// Dynamic programming stretegy.
// Input: Non negative integers n and k
//        such that $n \geq k$.
// Output: The value of $^nC_k$.
for i ← 0 to n do
    for j ← 0 to min(i, k) do
        if j = 0 or j = k
            C[i, j] ← 1
        else
            C[i, j] ← C{i-1, j-1] + C[i-1, j]
return C[n, k]
```

Analysis:

1. The complexity of the algorithm depends on n and k.
2. The basic operation is addition.

We can observe from the table that, for first k+1 rows, the entries will constitute a triangle. Afterwords, it will be of rectangle shape. So, the number of additions differ for first k+1 rows and for rest of n-k rows. Moreover, for each values of i and j is the algorithm, we have exactly one addition. Thus, the time complexity is given by

$$C = \sum_{i=1}^{k}\sum_{j=1}^{i-1} 1 + \sum_{i=k+1}^{n}\sum_{j=1}^{k} 1$$

$$= \sum_{i=1}^{k}(i-1-1+1) + \sum_{i=k+1}^{n}(k-1+1)$$

$$= \sum_{i=1}^{k}(i-1) + \sum_{i=k+1}^{n} k$$

$$= \sum_{i=1}^{k} i - \sum_{i=1}^{k} 1 + k.\sum_{i=k+1}^{n} 1$$

$$= \frac{k(k+1)}{2} - k + k(n-k-1+1)$$

$$= \frac{(k-1)k}{2} + k(n-k)$$

$$= \frac{k}{2}\{k-1+2n-2k\} = \frac{k(2n-k-1)}{2}$$

$$\simeq nk$$

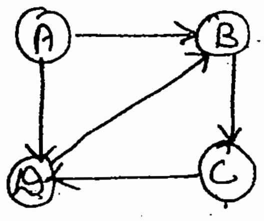
$$\therefore C(n,k) \in \Theta(nk)$$

Warshall's Algorithm

This algorithm is used for computing the transitive closure of a directed graph.

Def^n: The transitive closure of a directed graph with n vertices can be defined as the n x n boolean matrix $T = \{t_{ij}\}$ in which the element in the i^{th} row and j^{th} column is 1 if there exists a non-trivial directed path (i.e. a directed path of positive length) from i^{th} vertex to the j^{th} vertex, otherwise t_{ij} is 0.

For ex- consider ~~di~~digraph & its transitive closure



$$T = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 1 \\ B & 0 & 1 & 1 & 1 \\ C & 0 & 1 & 1 & 1 \\ D & 0 & 1 & 1 & 1 \end{array}$$

We can obtain the transitive closure by traversing the given digraph by either DFS or BFS. If we start any of these traversal at the i^{th} vertex, the vertices those can be reached from i can be found. So, by traversing the graph for all the vertices, we will get the transitive closure. But, the problem here is if the graph has n vertices, it must be traversed n times.

To overcome this problem, Warshall's algorithm is used, which will constructs the transitive closure of a given digraph with n vertices through a series of n x n boolean matrices $R^{(0)}, R^{(1)}, \ldots, R^{(k)} \ldots R^{(n)}$.

Each of these matrices will give some information on digraphs. The method for creating these matrices is given below-

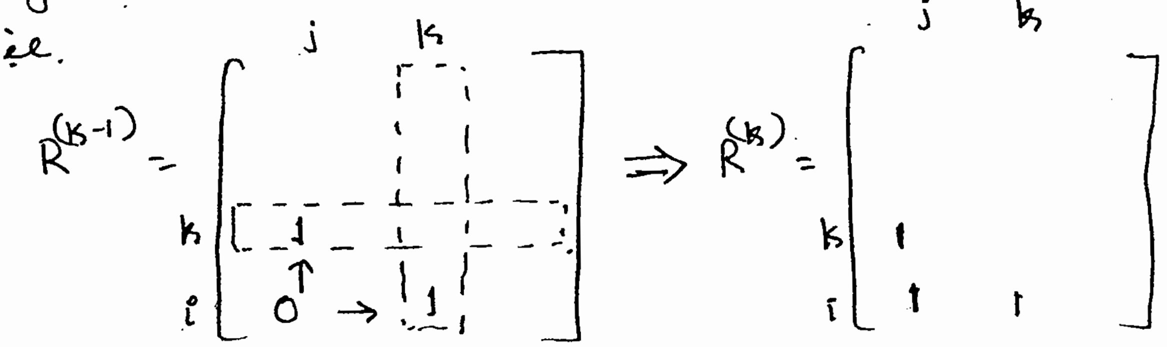
(i) $R^{(0)}$ is nothing but an adjacency matrix of the graph.

(ii) If an element r_{ij} is 1 in $R^{(k-1)}$, then it remains 1 in $R^{(k)}$.

(iii) If an element r_{ij} is 0 in $R^{(k-1)}$, it has to be changed to 1 in $R^{(k)}$ if and only if ~~the~~ $r_{ik} = r_{kj} = 1$ in $R^{(k-1)}$.

~~For illustration, consider the above graph. The series of matrices are given as below-~~

i.e.



For illustration, consider the graph given above. The series of matrices are shown below-

$$R^{(0)} = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 0 & 1 \\ B & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 1 \\ D & 0 & 1 & 0 & 0 \end{array}$$

This is nothing but the adjacency matrix of the given graph.

$$R^{(1)} = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 0 & 1 \\ B & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 1 \\ D & 0 & 1 & 0 & 0 \end{array}$$

NO new '1' is introduced here because, in $R^{(0)}$, $k=1$ & k^{th} column has only zeros.

$$R^{(2)} = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 1 \\ B & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 1 \\ D & 0 & 1 & 1 & 0 \end{array}$$

$r_{13}^{(2)}$ and $r_{43}^{(2)}$ are new '1's here.

$$R^{(3)} = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 1 \\ B & 0 & 0 & 1 & 1 \\ C & 0 & 0 & 0 & 1 \\ D & 0 & 1 & 1 & 1 \end{array}$$

$r_{24}^{(3)}$ & $r_{44}^{(3)}$ are newly introduced 1's.

$$R^{(4)} = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 1 \\ B & 0 & 1 & 1 & 1 \\ C & 0 & 1 & 1 & 1 \\ D & 0 & 1 & 1 & 1 \end{array}$$

$r_{22}^{(4)}$, $r_{32}^{(4)}$, $r_{33}^{(4)}$ are new 1's. This $R^{(4)}$ is transitive closure of given graph.

```
ALGORITHM Warshall(A[1..n][1..n])
// To implement Warshall's algorithm for
// computing the transitive closure.
// Input: The adjacency matrix A of a digraph
//            containing n vertices.
// Output: The transitive closure of the digraph.
$R^{(0)} \leftarrow A$
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            $R^{(k)}[i,j] \leftarrow R^{(k-1)}[i,j]$ or $R^{(k-1)}[i,k]$ and
                                                   $R^{(k-1)}[k,j]$
return $R^{(n)}$
```

<u>Analysis</u>: The basic operation is assignment based on 'or' and 'and' conditions. For each value of i, j and k, the operation is performed once. So,

$$C(n) = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} 1$$

$$= n^3$$

$$\therefore C(n) \in \Theta(n^3)$$

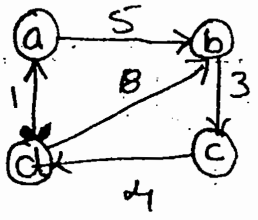
<u>NOTE</u>: The idea behind Warshall's algorithm has been generalised to get the shortest path between the vertices of a graph. This can be~~en~~ seen in Floyd's algorithm.

Floyd's Algorithm

Suppose that a weighted connected graph, that may be directed or undirected, is given. Finding the distances i.e. the length of shortest paths from each vertex to every other vertex of the graph is known as all-pairs shortest-paths problem. The distances from various vertices to other vertices are put in an n x n matrix format called the distance matrix D. For the weight (cost) matrix,

$$w_{ij} = \begin{cases} 0, & \text{if } i = j \\ \infty, & \text{if there is no edge between } i \,\&\, j \\ x, & \text{a positive value, if the distance between } i \,\&\, j \text{ is } x. \end{cases}$$

Consider a weighted digraph -



$$W = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 5 & \infty & \infty \\ b & \infty & 0 & 3 & \infty \\ c & \infty & \infty & 0 & 4 \\ d & 1 & 8 & \infty & 0 \end{array}$$

The distance matrix,

$$D = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 5 & 8 & 12 \\ b & 8 & 0 & 3 & 7 \\ c & 5 & 10 & 0 & 4 \\ d & 1 & 6 & 9 & 0 \end{array}$$

If the graph do not have a cycle of negative length, then the distance matrix can be generated by Floyd's algorithm. It suggests to generate a series of matrices $D^{(0)}, D^{(1)}, \ldots, D^{(k)}, \ldots D^{(n)}$ for a graph of n vertices based on the following rules-

$$d_{ij}^{(0)} = w_{ij} \quad \text{and}$$

$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\},\quad k \geq 1.$$

Following is the illustration for generating distance matrix by Floyd's algorithm for the above graph.

$$D^{(0)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 5 & \infty & \infty \\ b & \infty & 0 & 3 & \infty \\ c & \infty & \infty & 0 & 4 \\ d & 1 & 8 & \infty & 0 \end{array}$$

This is nothing but weight (cost) matrix.

$$D^{(1)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 5 & \infty & \infty \\ b & \infty & 0 & 3 & \infty \\ c & \infty & \infty & 0 & 4 \\ d & 1 & 6 & \infty & 0 \end{array}$$

~~Here $d_{42} = \min\{8, 6$~~

Here,

$$d_{42} = \min\{8,\ 1+5\}$$
$$= 6$$

$$D^{(2)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 5 & 8 & \infty \\ b & \infty & 0 & 3 & \infty \\ c & \infty & \infty & 0 & 4 \\ d & 1 & 6 & 9 & 0 \end{array}$$

$d_{13} = \min\{\infty, 8+3\}$
$= 8$

$d_{43} = \min\{\infty, 6+3\}$
$= 9$

$$D^{(3)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 5 & 8 & 12 \\ b & \infty & 0 & 3 & 7 \\ c & \infty & \infty & 0 & 4 \\ d & 1 & 6 & 9 & 0 \end{array}$$

$d_{24} = \min\{\infty, 3+4\}$
$= 7$

$d_{14} = \min\{\infty, 8+4\}$
$= 12$

$$D^{(4)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 5 & 8 & 12 \\ b & 8 & 0 & 3 & 7 \\ c & 5 & 10 & 0 & 4 \\ d & 1 & 6 & 9 & 0 \end{array}$$

$d_{21} = \min\{\infty, 1+7\}$
$= 8$

$d_{31} = \min\{\infty, 1+4\}$
$= 5$

$d_{32} = \min\{\infty, 6+4\}$
$= 10$

Here, $D^{(4)}$ is the require distance matrix.

```
ALGORITHM Floyd(W[1..n][1..n])
// Floyd's algo. for all-pair shortest-path problem
// Input: Weight matrix W of graph
// Output: Distance matrix D.
D ← W
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            D[i, j] ← min{D[i, j], D[i, k] + D[k, j]}
return D
```

The efficiency of the algorithm is -

$$C(n) = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} 1$$

$$= n^3$$

$$\therefore C(n) \in \Theta(n^3).$$

Knapsack Problem:-

Consider a knapsack problem of finding the most valueable subset of n items of weights $w_1, \dots, w_n$ and values $v_1, \dots, v_n$ that fit into a knapsack of capacity W. The dynamic programming strategy for solving this problem requires to derive a recurrence relation that expresses a solution to an instance of the knapsack problem interms of solutions to its smaller subinstances.

Consider an instance of a problem with first i items having weights $w_1, \dots, w_i$ and values $v_1, \dots, v_i$, and the knapsack of capacity j. Here, $1 \leq i \leq n$, $1 \leq j \leq W$. Let $V[i, j]$ be the most optimal solution for this instance. Now, the subsets of first i items that fit into knapsack of capacity j can be divided into two categories viz. those that do not contain i^{th} item and those they do contain i^{th} item. Then, we have the following -

* Among the subsets that do not have i^{th} item, the value of optimal solution is $V[i-1, j]$

* If the subsets include i^{th} item, then obviously $j - w_i \geq 0$. The optimal solution will be $v_i + V[i-1, j-w_i]$.

Thus, we have -

$$V[i, j] = \begin{cases} \max\{V[i-1, j], v_i + V[i-1, j-w_i]\}, & \text{if } j-w_i \geq 0 \\ V[i-1, j], & \text{if } j-w_i < 0 \end{cases}$$

with the initial conditions -

$V[0, j] = 0 \quad \forall j \geq 0$

$V[i, 0] = 0 \quad \forall i \geq 0$

Our requirement is to find $V[n, W]$ based on the above relation.

Ex: Consider the following problem with three items and the knapsack of capacity, $W = 4$. The weights and values as as -

Item	Weight	Value
A	3	25
B	1	20
C	2	40

Solution: Here, $W = 4$

$w_1 = 3$, $w_2 = 1$, $w_3 = 2$

$v_1 = 25$, $v_2 = 20$, $v_3 = 40$

As we know, $V[0, j] = 0 \quad \forall j \geq 0$

& $V[i, 0] = 0 \quad \forall i \geq 0$

Since there are only three items, the possibility for item inclusion may be one of 0, 1, 2, 3. Also, the capacity is 4. So, the possibility for one instance may be 0, 1, 2, 3 or 4. Thus, we have following table. The table entries are calculated as shown below.

			j		
i	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	25	25
2	0	20	20	25	45
3	0	20	40	60	60

When $i = 1$:—

$V[1,1] = V[i-1, j] = V[0,1] = 0$ $\quad \because j - w_i = 1-3 < 0$

$V[1,2] = V[i-1, j] = V[0,2] = 0$ $\quad \because j - w_i = -1 < 0$

$$V[1,3] = \max\{V[i-1,j],\ v_1 + V[i-1,\ j-w_i]\}$$
$$= \max\{V[0,3],\ 25 + V[0,0]\}$$
$$= \max\{0,\ 25+0\} = 25$$

$$V[1,4] = \max\{V[0,4],\ 25 + V[0,1]\}$$
$$= 25$$

When $i = 2$:—

$V[2,1] = \max\{V[1,1],\ 20 + V[1,0]\} = 20$

$V[2,2] = \max\{V[1,2],\ 20 + V[1,1]\} = 20$

$V[2,3] = \max\{V[1,3],\ 20 + V[1,2]\} = 25$

$V[2,4] = \max\{V[1,4],\ 20 + V[1,3]\} = 45$

When $i = 3$:—

$V[3,1] = V[2,1] = 20$ $\quad \because j - w_i < 0$

$V[3,2] = \max\{V[2,2],\ 40 + V[2,0]\} = 40$

$V[3,3] = \max\{V[2,3],\ 40 + V[2,1]\} = 60$

$V[3,4] = \max\{V[2,4],\ 40 + V[2,2]\} = 60$

After getting complete table, we have to look at the last entry. Because, we are interested in $V[n, W]$, where n is total number of items and W is total capacity.

Here, we have,

$V[n, W] = V[3, 4] = 60.$

So, the total profit we are going to get is 60.

Now we have to check, what are the items added to get this profit. Note that,

$V[2, 4] \neq V[3, 4].$ $(\because V(2,4] = 45$ & $V[3,4] = 60)$

This indicates, by inserting item 3 into the knapsack, we are gaining something. Thus, the item 3 must be one of our choices.

Now, from the problem table, it is seen that item 3 has weight 2. So, if we insert it, the remaining capacity is, $W - w_3 = 4 - 2$

$= 2$ only.

So, to check about next item, we have to look at the solution table only up to 2nd column. And, the row will be 2nd obviously, as we have considered item 3 already.

Now, $V(2, 2) \neq V[1, 2]$ $(\because V(1,2] = 0, V[2,2] = 20)$

This implies, inserting item 2, we are going to gain profit. So, item 2 must be a part of solution.

Now, $W - w_2 - w_3 = 4 - 1 - 2$

$= 1$

But, $w_1 = 3$. So, we can't insert item 1.

Thus solution is $\{2, 3\}$ or $\{B, C\}$ with the profit 60.

Memory Functions

The basic idea behind the dynamic programming is to reduce the number of calculations involved in top-down approach for ~~for~~ solving recurrence relations having overlapped subproblems. But for some problems, the dynamic programming strategy solves those subproblems that are not at all required for original problem. This is the disadvantage of bottom-up approach. So, to come out of these, we try to combine the good aspects of both top-down and bottom-up approaches. Such a method is based on memory functions.

In this method, we will solve the given problem in top-down approach only, but along with, we maintain the table structure as we do in usual bottom-up dynamic programming approach. Initially, all the entries of the table are initialized with a 'null' symbol to indicate that they have not yet been calculated. Then, whenever a new value has to be calculated, first the table is checked. If it is found, it is used.

Otherwise, if there is a 'null' value, it is calculated and stored at that position.

Thus, in this method, we calculate only those values which are required for the given problem.

~~Note that~~

Let us implement this method for knapsack problem. Here also, recursive relation remains same as before. i.e.

$$V[0,j] = 0 = V[i,0], \quad \forall i, j \geq 0$$

$$V[i,j] = \begin{cases} \max\{V[i-1,j],\ v_i + V[i-1, j-w_i]\}, & j-w_i \geq 0 \\ V[i-1,j], & \text{if } j-w_i < 0 \end{cases}$$

The algorithm for knapsack problem using memory function technique is given below-

```
ALGORITHM MFKnapsack(i, j)
// To implement memory function method for
// the knapsack problem.
// Input : i, indicating the number of items
//         j, indicating capacity, i, j ≥ 0
// Output : Value of optimal feasible subset of
//          first i items.
// NOTE : Use of global variables Wt[1..n],
//        Val[1..n] and V[0..n, 0..W] is made.
//        V[0..n, 0..W] is initialized with -1's
//        except row 0 & ~~[illegible]~~ column 0.
```

```
if V[i, j] < 0
    if j < Wt[i]
        val ← MFKnapsack(i-1, j)
    else
        val ← max (MFKnapsack(i-1, j),
                   Val[i] + MFKnapsack(i-1,
                                       j-Wt[i]))
    V[i, j] ← val
return V[i, j]
```

NOTE: To workout an example is leftout as an excertise for students!