Continuous Distributions

Continuous random variables are used to describe random phenomena in which variables take any value in some interval. Now, we will discuss various continuous distributions.

(i) Uniform Distribution

A random variable X is uniformly distributed on the interval $[a, b]$ if its pdf is given by-

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

The cdf is given by -

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \le x < b \\ 1, & \text{if } x \geq b \end{cases}$$

The mean and variance are:

$$E(X) = \frac{a+b}{2}$$

$$V(X) = \frac{(b-a)^2}{12}$$

NOTE: for uniform distribution,

$$P(x_1 < X < x_2) = F(x_2) - F(x_1)$$
$$= \frac{x_2 - x_1}{b-a}$$

Example:

A bus arrives every 20 minutes at a specified stop beginning at 6:40 am and continuing until 8:40 am. A certain passenger does not know the schedule, but arrives randomly (uniformly distributed) between 7:00 am and 7:30 am every morning. What is the probability that the passenger waits more than 5 minutes for a bus?

Solution: It is given that bus comes to the stop at every 20 min. starting from 6:40 am. Hence, bus would be coming at -

6:40 am, 7:00 am, 7:20 am, 7:40 am etc.

Passenger comes to bus stop anytime between 7 am and 7:30 am. The passenger has to wait for more than 5 minutes means -

* he would have arrived between 7:00 am and 7:15 am. That is he arrived after the 7 am bus left.

or * he would have arrived between 7:20 am and 7:30 am. That is, he came after 7:20 am bus left.

Let X denote the no of minutes past 7:00 am that the passenger arrives. Then, we need to compute.

$$P(0 < X < 15) + P(20 < X < 30)$$

Now, X is a random variable which follows uniform distribution in the interval (0, 30). So, the required probability can be presented using cdf as -

$$F(15) + F(30) - F(20)$$

Note that in this problem,

$a = 0$ and $b = 30$.

And x values are 15, 30 and 20 in cdf.

Thus,

$$F(15) + F(30) - F(20) = \frac{15-0}{30-0} + 1 - \frac{20-0}{30-0}$$

$$= \frac{1}{2} + 1 - \frac{2}{3}$$

$$= \frac{3+6-4}{6} = \frac{5}{6}$$

(ii) <u>Exponential Distribution</u>

A random variable X is said to be exponentially distributed with the parameter $\lambda > 0$, if its pdf is -

$$f(x) = \begin{cases} \lambda . e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The cdf is given by-

$$F(x) = \begin{cases} 0 & , \text{if } x < 0 \\ 1 - e^{-\lambda x} & , \text{if } x \geq 0 \end{cases}$$

The mean and variance are:

$$E(X) = \frac{1}{\lambda} \quad , \quad V(X) = 1/\lambda^2$$

<u>NOTE</u>: Exponential distribution is used mostly in following situations:

* To model inter-arrival time. Here λ would be arrival per hour.
* To model service time. Here, λ may be no. of services per minute.
* To model life time of a component that fails instantaneously. For ex. bulb. Here λ would be failure rate

Example:

Suppose, the life of an industrial lamp (calulated in thousands of hours) is exponentially distributed with failure rate $\lambda = 1/3$ (That is one failure for every 3000 hours, on an average). Compute (i) the probability that the life of a lamp is more than its mean life. i.e. 3000 hrs. (ii) Probability that life of lamp is between 2000 and 3000 hours.

Solution: (i) Let X be the no of hours (in thousands) denoting life of the lamp. Then, we need to compute $P(X > 3)$.

Now, $P(X>3) = 1 - P(X \le 3)$

$$= 1 - F(3)$$

$$= 1 - \left(1 - e^{-1/3 \cdot 3}\right)$$

$$= e^{-1}$$

$$= 0.3679$$

(ii) To find the probability of life of a lamp between 2000 to 3000 hrs. we need to compute $P(2 \le X \le 3)$.

Now,

$$P(2 \le X \le 3) = F(3) - F(2)$$

$$= \left(1 - e^{-1/3 \cdot 3}\right) - \left(1 - e^{-1/3 \cdot 2}\right)$$

$$= -e^{-1} + e^{-2/3}$$

$$= -0.3679 + 0.5134$$

$$= 0.1455$$

NOTE: Exponential distribution is "memoryless". To understand the meaning of this term, consider the following situation:

Let x be the life of a component. Now, the probability that the component lives for at least $s+t$ hours, given that it has already survived for s hours is given by-

$$P(x > s+t \mid x > s) = P(x > t)$$

Example:

Consider ~~the~~ the example of industrial lamp given in previous example. Compute the probability that the lamp will survive for another 1000 hours given that it has already survived 2500 hours.

Solution: As per the question, we need to compute.

$$P(x > 3.5 \mid x > 2.5) = P(x > 1)$$

$$= 1 - F(1)$$

$$= 1 - \left(1 - e^{-\frac{1}{3}\cdot 1}\right)$$

$$= e^{-1/3} = 0.7165$$

(ii) Normal Distribution

A random variable X is said to follow normal distribution with parameters μ and σ^2, if it has the pdf -

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

Here, $-\infty < \mu < \infty$ and $\sigma^2 > 0$.

The normal distribution is usually denoted by -

$$X \sim N(\mu, \sigma^2)$$

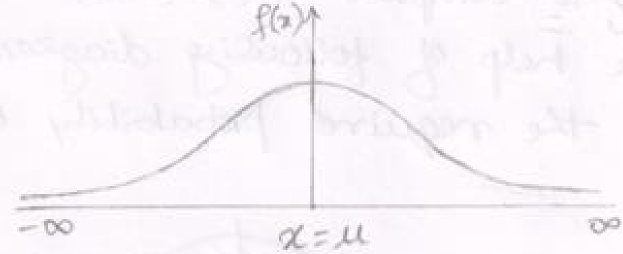
indicating X follows normal distribution. It has the following properties -

(i) $\lim\limits_{x \to -\infty} f(x) = 0 = \lim\limits_{x \to \infty} f(x)$

(ii) Normal pdf is symmetric about μ. That is -

$$f(\mu - x) = f(\mu + x)$$

The symmetry of normal distribution is shown using following diagram.



(iii) The maximum value of pdf occurs at $x = \mu$.

(iv) The mean and mode are equal for normal variate.

(v) If a random variable $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

That is, if x is a normal variate, then $\frac{x-\mu}{\sigma}$ is a normal variate with mean as zero and variance as 1. Now, Z is called as standard normal variate. The area under standard normal curve is 1.

NOTE: For solving many problems using normal distribution, we make use of standard normal variate and the pre-defined probability table is available for standard normal variate.

Example 1.

Let X be a normal variate such that,

$$X \sim N(50, 9).$$

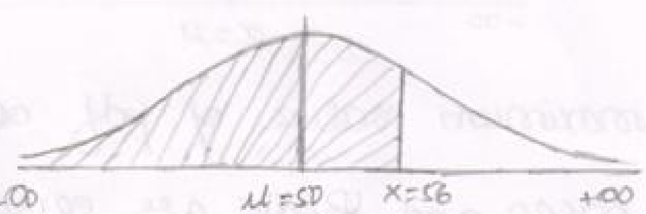
Then compute $P(X \leq 56)$.

Solution: It is given that $X \sim N(50, 9)$. That means,

$$\mu = 50 \quad \text{and} \quad \sigma^2 = 9$$

$$\Rightarrow \sigma = 3$$

We need to compute $P(X \leq 56)$. This can be understood with the help of following diagram. The shaded area indicates the required probability of $X \leq 56$.



Now, $Z = \frac{X-\mu}{\sigma} = \frac{56-50}{3} = 2 \sim N(0,1)$

Now, our requirement is $P(Z \leq 2)$. So, refer the standard normal distribution table (which is cumulative), and we get the value as 0.97725.

Thus, $P(X \leq 56) = 0.97725$

Example 2:

The time required to load an ocean-going vessel, X, is distributed as $N(12, 4)$. Compute the probability that the vessel will be loaded

(i) in less than 10 hours

(ii) more than 12 hours

(iii) inbetween 10 and 12 hours.

Solution: It is given that,

$X \sim N(12, 4)$.

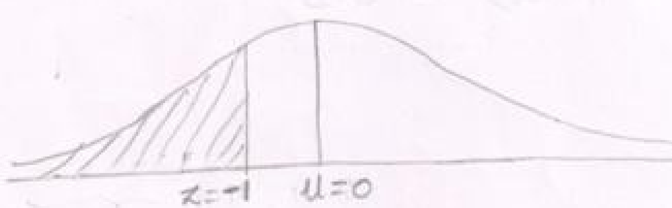
That is, $\mu = 12$ and $\sigma^2 = 4$.

Now, $Z = \frac{X - 12}{2} \sim N(0, 1)$

(i) To compute the probability that the vessel will be loaded in less than 10 hours:

$P(X \leq 12) \Rightarrow Z = \frac{10 - 12}{2} = -1 \sim N(0, 1)$.

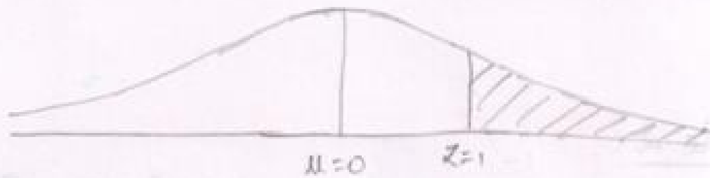
The standard normal curve would be —



We know that normal distribution is symmetric. So,

$P(X \le t) = 1 - P(X \geq t)$.

That is, in this example, we can have required curve as-



Using cumulative standard normal table, we will get-

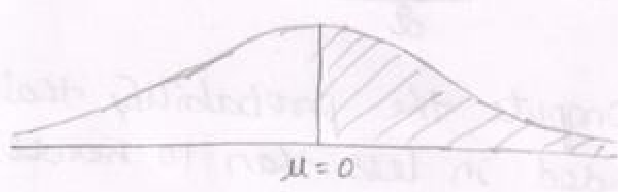
$$P(Z \le -1) = 1 - P(Z \geq 1)$$
$$= 1 - 0.84134$$
$$= 0.15866$$

(ii) More than 12 hours -

$$P(X \geq 12) \Rightarrow Z = \frac{12-12}{2} = 0$$

i.e. we need to compute, $P(Z \geq 0)$.

The curve would be -

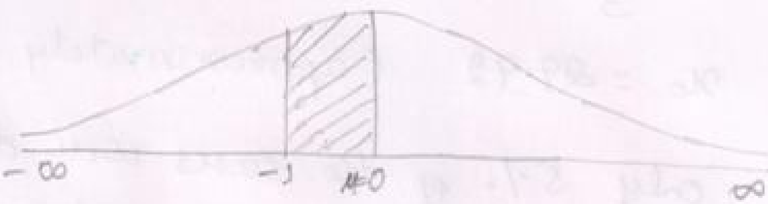


Without referring to table, just by intuition, we can say that the required probability is 0.50.

That is -

$$P(X \geq 12) = 0.5$$

(iii) vessel is loaded between 10 and 12 hours. We need to compute $P(10 \le X \le 12)$. Using standard normal variate, it would be $P(-1 \le Z \le 0)$. The curve is –



Now using symmetric nature of normal distribution we say that required probability will be –

(area from $-\infty$ to 0) – (area from $-\infty$ to -1)

$= 0.5 - 0.15866$

$= 0.34134$

Example 3:

Lead-time demand, X, for an item has distribution $N(25, 9)$. Compute the value of leadtime that will be exceeded only 5% of the time.

Solution: Here, we need to find some value x_0 such that $P(X > x_0) = 0.05$.

i.e. $P(X > x_0) = P\left(Z > \frac{x_0 - 25}{3}\right)$

$= 1 - P\left(\frac{x_0 - 25}{3}\right) = 0.05$

$\Rightarrow P\left(Z < \frac{x_0 - 25}{3}\right) = 0.95.$

Now, search the cumulative standard normal table for the value 0.95. It is found at 1.64.

Thus $P(z < 1.64) = 0.95$

$$\Rightarrow \frac{x_0 - 25}{3} = 1.64$$

$$\Rightarrow x_0 = 29.92 \quad (\text{approximately } 30)$$

Thus, only 5% of the cases will demand during lead-time exceed available inventory, if the order to purchase is made when the stock reaches 30.

Gamma Distribution

Gamma distribution of a random variable X is defined using a Gamma function. A gamma function for any $\beta > 0$ is given as —

$$\Gamma\beta = \int_0^\infty x^{\beta-1} \cdot e^{-x}\, dx$$

Note that $\Gamma 1 = 1$ and $\Gamma\beta = (\beta - 1)!$ when β is positive integer

A random variable X is said to follow gamma distribution with parameters β and θ, if its pdf is —

$$f(x) = \begin{cases} \frac{\beta\theta}{\Gamma\beta} (\beta\theta x)^{\beta-1} \cdot e^{-\beta\theta x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

In gamma distribution, the parameter β is called as shape parameter and θ is called as scale parameter.

The mean and variance are given by -

$$E(x) = \frac{1}{\theta} \quad \text{and} \quad V(x) = \frac{1}{\beta\theta^2}$$

The cdf of x is given by -

$$F(x) = \begin{cases} 1 - \int_x^\infty \frac{\beta\theta}{\Gamma\beta} \cdot (\beta\theta t)^{\beta-1} \cdot e^{-\beta\theta t} \cdot dt, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

Beta Distribution

A random variable X is beta distributed with parameters $\beta 1 > 0$ and $\beta 2 > 0$, if the pdf is -

$$f(x) = \begin{cases} \frac{x^{\beta_1 - 1} \cdot (1-x)^{\beta_2 - 1}}{B(\beta 1, \beta 2)}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Here,

$$B(\beta_1, \beta_2) = \frac{\Gamma\beta_1 \cdot \Gamma\beta_2}{\Gamma\beta_1 + \beta_2}$$

The pdf, $f(x)$, of beta distribution given above is defined in the range of (0, 1). But, in general, distribution is defined in the range (a, b), where $a < b$. Then, we can define a new random variable as –

$$Y = a + (b-a)X.$$

Now, mean and variance of Y are given as –

$$E(Y) = a + (b-a)\left(\frac{\beta_1}{\beta_1 + \beta_2}\right)$$

and -

$$V(Y) = (b-a)^2\left(\frac{\beta_1 \beta_2}{(\beta_1 + \beta_2)^2 (\beta_1 + \beta_2 + 1)}\right)$$

Poisson Process

Consider random events such as-

* arrival of jobs at a job shop
* arrival of email to a mail server
* arrival of boats to a dock
* arrival of calls to a call center
* breakdown of machines in a large factory etc.

These events are described by a counting function $N(t)$ defined for all $t \geq 0$. This counting function will represent the no of events that occurred in $[0, t]$. Here, the time zero indicates the time at which the observation began. For each interval $[0, t]$, the value $N(t)$ is an observation of a random variable which takes the integer values 0, 1, 2, ...

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with mean λ if the following assumptions are satisfied:

(i) Arrivals occur one at a time.

(ii) $\{N(t), t \geq 0\}$ has stationary increments: The distribution of the no of arrivals between t and $t+s$ depends only on the length of the interval s, not on the starting point t. Thus arrivals are completely at random without rush or slack periods.

(iii) $\{N(t), t \geq 0\}$ has independent increments: The no of arrivals during non-overlapping time intervals are independent random variables.

If arrivals according to a Poisson process, satisfying above three assumptions, it can be shown that the probability that $N(t)$ is equal to n is given by—

$$P[N(t) = n] = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots \quad t \geq 0$$

Thus, $N(t)$ has the poisson distribution with parameter $\alpha = \lambda t$. Thus, mean and variance are—

$$E(N(t)) = \lambda t$$

$$V(N(t)) = \lambda t$$

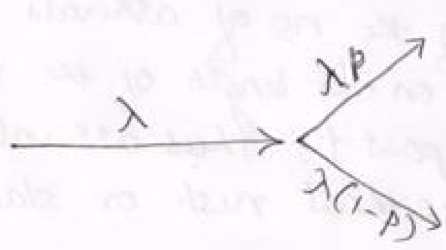
Properties of a Poisson Process

The two properties of a poisson process which are useful in discrete system simulation are—

(i) Random Splitting

(ii) Pooled Process

(i) Random Splitting: Consider a poisson process $\{N(t), t \geq 0\}$ having rate λ as shown in following figure—



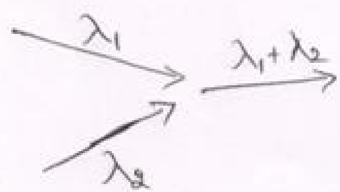
Suppose, each event is classified as type I event with probability p and type II event with probability $1-p$. Let $N_1(t)$ and $N_2(t)$ be the random variables denoting no of type I and type II events respectively.

Note that the events are occuring in $[0, t]$ and,

$$N(t) = N_1(t) + N_2(t)$$

The two events $N_1(t)$ and $N_2(t)$ are independent Poisson processes having rates λp and $\lambda(1-p)$.

(ii) <u>Pooled</u> <u>Process</u> : Consider the opposite situation of random splitting, the pooling of two arrival streams as shown –

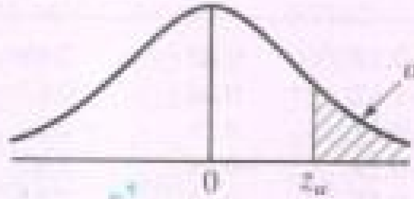


If $N_1(t)$ and $N_2(t)$ are two random variables representing independent Poisson processes with rates λ_1 and λ_2, then

$$N(t) = N_1(t) + N_2(t)$$ is a Poisson process

with rate $\lambda_1 + \lambda_2$.

Table A.3 Cumulative Normal Distribution



$$\phi(z_\alpha) = \int_{-\infty}^{z_\alpha} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du = 1 - \alpha$$

z_α	0.00	0.01	0.02	0.03	0.04	z_α
0.0	0.500 00	0.503 99	0.507 98	0.511 97	0.515 95	**0.0**
0.1	0.539 83	0.543 79	0.547 76	0.551 72	0.555 67	**0.1**
0.2	0.579 26	0.583 17	0.587 06	0.590 95	0.594 83	**0.2**
0.3	0.617 91	0.621 72	0.625 51	0.629 30	0.633 07	**0.3**
0.4	0.655 42	0.659 10	0.662 76	0.666 40	0.670 03	**0.4**
0.5	0.691 46	0.694 97	0.698 47	0.701 94	0.705 40	**0.5**
0.6	0.725 75	0.729 07	0.732 37	0.735 65	0.738 91	**0.6**
0.7	0.758 03	0.761 15	0.764 24	0.767 30	0.770 35	**0.7**
0.8	0.788 14	0.791 03	0.793 89	0.796 73	0.799 54	**0.8**
0.9	0.815 94	0.818 59	0.821 21	0.823 81	0.826 39	**0.9**
1.0	0.841 34	0.843 75	0.846 13	0.848 49	0.850 83	**1.0**
1.1	0.864 33	0.866 50	0.868 64	0.870 76	0.872 85	**1.1**
1.2	0.884 93	0.886 86	0.888 77	0.890 65	0.892 51	**1.2**
1.3	0.903 20	0.904 90	0.906 58	0.908 24	0.909 88	**1.3**
1.4	0.919 24	0.920 73	0.922 19	0.923 64	0.925 06	**1.4**
1.5	0.933 19	0.934 48	0.935 74	0.936 99	0.938 22	**1.5**
1.6	0.945 20	0.946 30	0.947 38	0.948 45	0.949 50	**1.6**
1.7	0.955 43	0.956 37	0.957 28	0.958 18	0.959 07	**1.7**
1.8	0.964 07	0.964 85	0.965 62	0.966 37	0.967 11	**1.8**
1.9	0.971 28	0.971 93	0.972 57	0.973 20	0.973 81	**1.9**
2.0	0.977 25	0.977 78	0.978 31	0.978 82	0.979 32	**2.0**
2.1	0.982 14	0.982 57	0.983 00	0.983 41	0.983 82	**2.1**
2.2	0.986 10	0.986 45	0.986 79	0.987 13	0.987 45	**2.2**
2.3	0.989 28	0.989 56	0.989 83	0.990 10	0.990 36	**2.3**
2.4	0.991 80	0.992 02	0.992 24	0.992 45	0.992 66	**2.4**
2.5	0.993 79	0.993 96	0.994 13	0.994 30	0.994 46	**2.5**
2.6	0.995 34	0.995 47	0.995 60	0.995 73	0.995 85	**2.6**
2.7	0.996 53	0.996 64	0.996 74	0.996 83	0.996 93	**2.7**
2.8	0.997 44	0.997 52	0.997 60	0.997 67	0.997 74	**2.8**
2.9	0.998 13	0.998 19	0.998 25	0.998 31	0.998 36	**2.9**
3.0	0.998 65	0.998 69	0.998 74	0.998 78	0.998 82	**3.0**
3.1	0.999 03	0.999 06	0.999 10	0.999 13	0.999 16	**3.1**
3.2	0.999 31	0.999 34	0.999 36	0.999 38	0.999 40	**3.2**
3.3	0.999 52	0.999 53	0.999 55	0.999 57	0.999 58	**3.3**
3.4	0.999 66	0.999 68	0.999 69	0.999 70	0.999 71	**3.4**
3.5	0.999 77	0.999 78	0.999 78	0.999 79	0.999 80	**3.5**
3.6	0.999 84	0.999 85	0.999 85	0.999 86	0.999 86	**3.6**
3.7	0.999 89	0.999 90	0.999 90	0.999 90	0.999 91	**3.7**
3.8	0.999 93	0.999 93	0.999 93	0.999 94	0.999 94	**3.8**
3.9	0.999 95	0.999 95	0.999 96	0.999 96	0.999 96	**3.9**

continues...

Table A.3 Continued

z_α	0.05	0.06	0.07	0.08	0.09	z_α
0.0	0.519 94	0.523 92	0.527 90	0.531 88	0.535 86	**0.0**
0.1	0.559 62	0.563 56	0.567 49	0.571 42	0.575 34	**0.1**
0.2	0.598 71	0.602 57	0.606 42	0.610 26	0.614 09	**0.2**
0.3	0.636 83	0.640 58	0.644 31	0.648 03	0.651 73	**0.3**
0.4	0.673 64	0.677 24	0.680 82	0.684 38	0.687 93	**0.4**
0.5	0.708 84	0.712 26	0.715 66	0.719 04	0.722 40	**0.5**
0.6	0.742 15	0.745 37	0.748 57	0.751 75	0.754 90	**0.6**
0.7	0.773 37	0.776 37	0.779 35	0.782 30	0.785 23	**0.7**
0.8	0.802 34	0.805 10	0.807 85	0.810 57	0.813 27	**0.8**
0.9	0.824 94	0.831 47	0.833 97	0.836 46	0.838 91	**0.9**
1.0	0.853 14	0.855 43	0.857 69	0.859 93	0.862 14	**1.0**
1.1	0.874 93	0.876 97	0.879 00	0.881 00	0.882 97	**1.1**
1.2	0.894 35	0.896 16	0.897 96	0.899 73	0.901 47	**1.2**
1.3	0.911 49	0.913 08	0.914 65	0.916 21	0.917 73	**1.3**
1.4	0.926 47	0.927 85	0.929 22	0.930 56	0.931 89	**1.4**
1.5	0.939 43	0.940 62	0.941 79	0.942 95	0.944 08	**1.5**
1.6	0.950 53	0.951 54	0.952 54	0.953 52	0.954 48	**1.6**
1.7	0.959 94	0.960 80	0.961 64	0.962 46	0.963 27	**1.7**
1.8	0.967 84	0.968 56	0.969 26	0.969 95	0.970 62	**1.8**
1.9	0.974 41	0.975 00	0.975 58	0.976 15	0.976 70	**1.9**
2.0	0.979 82	0.980 30	0.980 77	0.981 24	0.981 69	**2.0**
2.1	0.984 22	0.984 61	0.985 00	0.985 37	0.985 74	**2.1**
2.2	0.987 78	0.988 09	0.988 40	0.988 70	0.988 99	**2.2**
2.3	0.990 61	0.990 86	0.991 11	0.991 34	0.991 58	**2.3**
2.4	0.992 86	0.993 05	0.993 24	0.993 43	0.993 61	**2.4**
2.5	0.994 61	0.994 77	0.994 92	0.995 06	0.995 20	**2.5**
2.6	0.995 98	0.996 09	0.996 21	0.996 32	0.996 43	**2.6**
2.7	0.997 02	0.997 11	0.997 20	0.997 28	0.997 36	**2.7**
2.8	0.997 81	0.997 88	0.997 95	0.998 01	0.998 07	**2.8**
2.9	0.998 41	0.998 46	0.998 51	0.998 56	0.998 61	**2.9**
3.0	0.998 86	0.998 89	0.998 93	0.998 97	0.999 00	**3.0**
3.1	0.999 18	0.999 21	0.999 24	0.999 26	0.999 29	**3.1**
3.2	0.999 42	0.999 44	0.999 46	0.999 48	0.999 50	**3.2**
3.3	0.999 60	0.999 61	0.999 62	0.999 64	0.999 65	**3.3**
3.4	0.999 72	0.999 73	0.999 74	0.999 75	0.999 76	**3.4**
3.5	0.999 81	0.999 81	0.999 82	0.999 83	0.999 83	**3.5**
3.6	0.999 87	0.999 87	0.999 88	0.999 88	0.999 89	**3.6**
3.7	0.999 91	0.999 92	0.999 92	0.999 92	0.999 92	**3.7**
3.8	0.999 94	0.999 94	0.999 95	0.999 95	0.999 95	**3.8**
3.9	0.999 96	0.999 96	0.999 96	0.999 97	0.999 97	**3.9**