Random-Variate Generation

When probability distributions are known, it is possible to generate random variates. The random variates are then used as input to a simulation model. Here, we will discuss inverse-transform technique and acceptance-rejection technique for generating random variates. And, we assume that random numbers are readily available from uniform distribution in the range [0, 1].

Inverse-Transform Technique:

Inverse transform technique can be used on uniform, exponential, Weibull and traingular distributions. It is useful when the cdf $F(x)$ is very simple and F^{-1} (meaning is F-inverse) is easy to calculate. The steps are given below–

<u>step 1</u>: Compute the cdf of desired random variable X.

<u>step 2</u>: Set $F(x) = R$ on the range of X.

<u>step 3</u>: Solve the equation $F(x) = R$, for X in terms of R. That is,

$$X = F^{-1}(R)$$

<u>step 4</u>: Generate uniform random numbers $R_1, R_2, R_3 \cdots$ by putting

$$X_i = F^{-1}(R_i)$$

Inverse transform Technique for Exponential Distribution

The exponential distribution has pdf –

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

The cdf is given by –

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

The purpose is to generate $X_1, X_2, \cdots$ that have exponential distribution. The procedure is as follows –

step 1: Compute cdf. i.e.

$$F(x) = 1 - e^{-\lambda x}$$

step 2: Set $F(X) = R$

$$\Rightarrow \quad 1 - e^{-\lambda X} = R$$

step 3: Solve $F(X) = R$ in terms of X.

i.e. $\quad 1 - e^{-\lambda X} = R$

$$\Rightarrow \quad e^{-\lambda X} = 1 - R$$

$$\Rightarrow \quad -\lambda X = \ln(1-R) \qquad [\text{Note: } \ln(P) = \log_e(P)]$$

$$\Rightarrow \quad X = -\frac{1}{\lambda} \cdot \ln(1-R)$$

This also can be written as $X = F^{-1}(R)$

step 4: Generate $R_1, R_2, R_3 \ldots\ldots$ and compute X_i using

$$X_i = F^{-1}(R_i)$$

[As R_i is uniformly distributed, $1-R_i$ can be replaced by R_i. ie,

$$X_i = -\frac{1}{\lambda} \ln(R_i)$$

Using above formula also, one can generate random variates]

Example: Generate exponential random variates with $\lambda = 1$. Given random numbers are: 0.1306, 0.0422, 0.6597, 0.7965, 0.7696.

Solution: Given that $\lambda = 1$.

We know that, random variates are generated using random numbers with help of formula—

$$X_i = -\frac{1}{\lambda} \cdot \ln(1-R_i)$$

$$\Rightarrow X_i = -\ln(1-R_i)$$

(Note that $\ln(R_i)$ is nothing but $\log_e(R_i)$)

The random numbers and respective random variates are given below—

i	1	2	3	4	5
R_i	0.1306	0.0422	0.6597	0.7965	0.7696
X_i	0.1399	0.0431	1.0779	1.5921	1.4679

Inverse transform Technique for Uniform Distribution

A random variable X having uniform distribution in the interval $[a, b]$ has the pdf —

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

The cdf is given by—

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x > b \end{cases}$$

To apply inverse transform technique to generate random variates which follows uniform distribution, following steps are applied.

Step 1. The cdf is.

$$F(x) = \frac{x-a}{b-a}$$

Step 2. Set $F(x) = R$

$$\Rightarrow \frac{x-a}{b-a} = R$$

Step 3: Solve for x.

$$\Rightarrow x - a = (b-a)R$$

$$\Rightarrow X = a + (b-a)R$$

Step 4: Given a set of random numbers R_i, compute x_i using the formula—

$$X_i = a + (b-a)R_i$$

Example: Generate random variates that follows uniform distribution in the interval [3, 5]. Given the random numbers: 0.779, 0.921, 0.186, 0.289, 0.653.

Solution: Uniform distribution is in [3, 5]. That is, $a = 3$ and $b = 5$.

For a given set of random numbers, the random variates are generated using the formula –

$$X_i = a + (b-a) R_i$$

The random numbers and respective random variates are given below –

R_i	0.779	0.921	0.186	0.289	0.653
X_i	4.558	4.842	3.372	3.578	4.306

Acceptance - Rejection Technique:

One more method for generating random variates is acceptance-rejection technique. Here, every random number is checked whether it satisfies some condition. If so, the random variate is accepted. Otherwise, the random variate is rejected. When new random variate is required, new random number is generated and procedure is repeated.

The procedure of acceptance-rejection technique for generating random variates of uniform distribution is illustrated here under:–

Suppose an analyst needs to devise a method for generating random variates X, uniformly distributed between $\frac{1}{4}$ and 1. The acceptance-rejection technique for this situation has following steps:

<u>step 1</u>: Generate a random number R.

<u>step 2a</u>: If $R \geq 1/4$, accept $X = R$, then goto step 3.

<u>step 2b</u>: If $R < \frac{1}{4}$, reject R, and return to step 1.

<u>step 3</u>: If another uniform random variate on $[\frac{1}{4}, 1]$ is needed, repeat the procedure from step 1. Otherwise, stop.

<u>NOTE</u>: The efficiency of acceptance-rejection technique depends on being able to minimize the number of rejections.

<u>Acceptance-Rejection technique for Poisson Distribution</u>

A poisson random variable N with mean $\alpha > 0$ has the pmf –

$$p(n) = P(N = n) = \frac{e^{-\alpha}\alpha^n}{n!}, \quad n = 0, 1, 2, \dots$$

Here, N can also be thought of number of arrivals from a Poisson arrival process in one unit of time.

The procedure for generating a Poisson random variate, N is given by following steps –

step 1: Set $n = 0$, $P = 1$

step 2: Generate a random number R_{n+1} and replace P by $P \cdot R_{n+1}$. That is, $P = P \cdot R_{n+1}$

step 3: If $P < e^{-\alpha}$, then accept $N = n$. Otherwise, reject the current n, increase n by one and return to step 2.

Example: Using the acceptance-rejection technique, generate three Poisson variates with mean $\alpha = 0.2$. Given random numbers are: 0.4357, 0.4146, 0.8353, 0.9952, 0.8004.

Solution: Given that $\alpha = 0.2$. The procedure is as follows:

step 1: Set $n = 0$, $P = 1$.

step 2: $R_{n+1} = R_1 = 0.4357$

Now, $P = P \cdot R_{n+1}$

$= 1 \cdot 0.4357$

$\Rightarrow P = 0.4357$

step 3: Check whether $P < e^{-\alpha}$

i.e. $0.4357 < e^{-0.2}$ is true?

$0.4357 < 0.8187$ is true. Hence, accept $N = n$. That is $n = 0$ is first random variate.

Since, we require 3 random variates, we restart with step 1 again.

<u>step 1</u>: Set $n=0$, $P=1$

<u>step 2</u>: Now, $R_{n+1} = R_1 = 0.4146$.

$P = P \cdot R_{n+1} = 0.4146$.

<u>step 3</u>: As $P < e^{-\alpha}$

i.e. $0.4146 < 0.8187$, we accept $N=n$,

i.e. $N=0$ is accepted again.

Hence, 2nd random variate is also $N=0$.

For 3rd random variates, again start from step 1.

<u>step 1</u>: Set $n=0$, $P=1$.

<u>step 2</u>: Now, $R_{n+1} = R_1 = 0.8353$.

$P = P \cdot R_1 = 0.8353$.

<u>step 3</u>: Now, $P < e^{-\alpha}$ is false, because,

$0.8353 \not< 0.8187$.

Hence $N=0$ is rejected.

Now, $n = n+1 = 1$. Go to step 2.

<u>step 2</u>: $R_{n+1} = R_2 = 0.9952$.

$P = P \cdot R_2 = (0.8353)(0.9952)$

$= 0.8313$

<u>step 3</u>: Now, $P < e^{-\alpha}$ is false again, as—

$0.8313 \not< 0.8187$.

So, $N=1$ is also rejected.

Now, $n = n+1 = 2$. Go to step 2.

<u>step 2</u>: $R_{n+1} = R_3 = 0.8004$.

$P = P \cdot R_3 = (0.8313)(0.8004) = 0.6654$

Step3 : Now $P < e^{-\alpha}$ is true, as

$0.6654 < 0.8187$

Hence $N = 2$ is accepted.

Thus, the required random variates are –

0, 0, 2.

NOTE: The calculation of above problem can be summerized in the following table –

n	R_{n+1}	P	Accept/Reject	Result
0	0.4357	0.4357	$P < e^{-\alpha}$ (accept)	N = 0
0	0.4146	0.4146	$P < e^{-\alpha}$ (accept)	N = 0
0	0.8353	0.8353	$P > e^{-\alpha}$ (reject)	
1	0.9952	0.8313	$P > e^{-\alpha}$ (reject)	
2	0.8004	0.6654	$P < e^{-\alpha}$ (accept)	N = 2

Example: Buses arrive at the bus stop according to a Poisson process with a mean of one bus per 15 minutes. Generate a random variate N which represents the number of arriving buses during 1-hour time slot.

Solution: As per the data given, we can say that average number of buses per hour = 4.

i.e. $\alpha = 4$

and $e^{-\alpha} = e^{-4} = 0.0183$

For solving this problem, random numbers are taken from pre-defined random number table.

The series of steps are summerized in the table given below –

n	R_{n+1}	P	Accept or Reject	Result.
0	0.4357	0.4357	$P > e^{-\alpha}$ (reject)	
1	0.4146	0.1806	$P > e^{-\alpha}$ (reject)	
2	0.8353	0.1509	$P > e^{-\alpha}$ (reject)	
3	0.9952	0.1502	$P > e^{-\alpha}$ (reject)	
4	0.8004	0.1202	$P > e^{-\alpha}$ (reject)	
5	0.7945	0.0955	$P > e^{-\alpha}$ (reject)	
6	0.1530	0.0146	$P < e^{-\alpha}$ (accept)	$N = 6$

Thus, $N = 6$ is the no of arriving buses during 1-hour time slot.

<u>NOTE</u>: If x number of random variates have to be generated, then $x(\alpha + 1)$ number of random numbers are required, in general.