# UNIT 4. QUEUING MODELS

Simulation is frequently used in the analysis of queuing models. In a typical queuing model (shown in Figure 4.1), customers arrive from time to time and join a queue (waiting line). They are eventually served and later, leave the system.
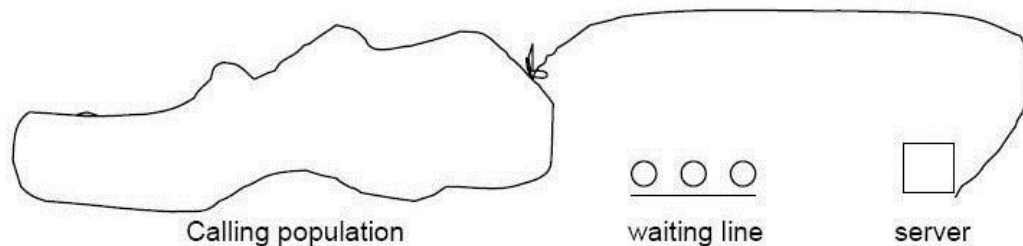


Figure 4.1 Simple Queuing Model

Queuing models provide a powerful tool for designing and evaluating the performance of queuing systems. System performance may be measured with the help of server utilization, length of waiting lines, delays of customers etc. The analyst has to tradeoff these measures to take proper decision according to the situation. Queuing theory and simulation analysis are used to predict these measures of system performances as a function of input parameters. The input parameters may be – arrival rate of customers, service demands of customers, rate at which a server works, number and arrangement of servers etc.

## 4.1  CHARACTERISTICS OF QUEUING SYSTEMS
Major components of a queuing system are the customers and servers. The term *customer* can refer to people, machines, trucks, mechanics, patients, pallets, airplanes, email, cases, orders etc – anything that arrives at a facility and requires service. The term *server* may refer to receptionists, repair personnel, mechanics, medical personnel, retrieval machines like cranes, runways at airport, order pickers, CPU of computer, washing machines etc – any resource that provides service.  In general, we say that the customer goes to a service facility. But, there are exceptions – a plumber/electrician goes to house for the repair work.

Various elements of a queuing system are discussed hereunder.

### 4.1.1  The Calling Population
The set of potential customers is referred as *calling population.*  It may be finite or infinite. For example, the computers at a small company are customers and can be treated as finite population. The technicians who resolve the problem of these computers can be treated as servers. People arriving at a bank or a hotel can be considered as infinite calling population, as there is no exact count of people visiting every day.

The major difference between finite and infinite population models is how the arrival rate is defined. Arrival rate is the average number of arrivals per unit of time. In an infinite population model, the arrival rate is not affected by the number of customers who have left the calling population and joined the queuing system. When the arrival process is

homogeneous over time, that is, if there are no rush hours, then the arrival time is assumed to be constant. On the other hand, for finite calling population, the arrival time depends on number of customers. For example, assume that there are 5 patients who have to be attended by one nurse. When all patients are taking rest, the nurse (server) would be idle. Moreover, in the very next moment nurse may get a call from all 5 patients, out of which, she can attend only one at a time. So, now 4 patients will be in a waiting queue. Thus, arrival rate is maximum when the server is idle, because, expectation of number of calls in the very next moment is high.

### 4.1.2 System Capacity

In many queuing systems, there is a limit for the number of customers that can be in a waiting line. For example, in a car-wash center, there may be a place for 10 cars. When these 10 cars are waiting for their turn, another car may come for getting the service. But, since there is no place, it has to go back and we say that, it has entered the calling population again. When the system has limited capacity, a distinction is made between the arrival rate (the number of arrivals per time unit) and the effective arrival rate (the number who arrive and enter the system per time unit).

### 4.1.3 The Arrival Process

The arrival process for infinite population models is characterized in terms of inter-arrival times of successive customers. Arrivals may occur at scheduled times or at random times. When it is random, inter-arrival times are characterized by probability distribution. Moreover, the customers may arrive one at a time or in batches. Also, the batch size may be constant or of random.

If $A_n$ represents the inter-arrival time between $(n-1)^{th}$ customer and $n^{th}$ customer, then $A_n$ is exponentially distributed with mean $1/$ time units. The arrival rate = number of customers per time unit. The number of arrivals in a time interval of length $t$, say, *N(t)* has the Poisson distribution with mean $t$ customers.

### 4.1.4 Queue Behavior and Queue Discipline

Queue behavior refers to the actions of customers while in a queue waiting for service to begin. Few of the behaviors are listed as below –
*   **Balk:** Leave when they see that the line is too long
*   **Renege:** Leaver after being in the line when they see that the line is moving too slowly.
*   **Jockey:** Move from one line to another if they think that they have chosen a slow line.

Queue discipline refers to the logical ordering of customers in a queue and determines which customer will be chosen next for the service. Common queue disciplines are –
*   First In First Out (FIFO)
*   Last In First Out (LIFO)
*   Service in Random Order (SIRO)
*   Shortest Processing Time First (SPT)
*   Service according to Priority (PR)

### 4.1.5 Service Times and the Service Mechanism

The service times of successive arrivals are denoted by S1, S2, S3, … and they may be of constant or of random duration. If they are random, the set {S1, S2, S3, …} is usually characterized as a sequence of independent and identically distributed random variables.

A queuing system consists of a number of service centers and interconnecting queues. Each service center consists of some number of servers, *c*, working in parallel. Parallel service mechanisms are either singer server (*c=1*), multiple server (1<c< ) or unlimited servers (c= ). A self service facility is usually characterized as having an unlimited number of servers.

## 4.2 QUEUING NOTATIONS

The queuing system is denoted in the format A/B/c/N/K. The meaning of these letters is as listed –
- o **A** : Inter-arrival time distribution
- o **B** : service-time distribution
- o **c** : number of parallel servers
- o **N** : System capacity
- o **K** : Size of the calling population

Here, A and B may be one of the following symbols –
- **M**: Markov or Exponential
- **D:** Constant or Deterministic
- $E_k$ : Erlang of order *k*
- **PH:** Phase-type
- **H :** Hyper-exponential
- **G:** General or arbitrary
- **GI :** General independent

**Example 1:** The notation M/M/1/ / indicates a single-server system that has unlimited queue capacity and an infinite population of potential arrivals. The inter-arrival time and service times are exponentially distributed. Note that, when N and K are infinite they can be dropped from the notation. That is, M/M/1/ / can be written as M/M/1.

**Example 2:** One nurse attending 5 patients in a hospital can be represented as M/M/1/5/5.

## 4.3 SIMULATION EXAMPLE: QUEUING

Here, we initially discuss the components of queuing or waiting line models. Then two dynamic, event-based models viz single–server queue and two–server queue are discussed with the help of examples.

### 4.3.1 Waiting Line Models

The queuing system is described by few attributes like calling population, nature of arrivals, service mechanism, system capacity and the queuing discipline. The members of the calling population may be customers, telephone calls etc, which we refer as *units.*
Certain assumptions are made here –
- Calling population is infinite
- System capacity is infinite

- Arrival process is random with certain probability distribution
- Queue discipline is FIFO
- Service times are random variables with certain probability distribution
- Arrival rate is less than service rate (otherwise, queue will grow without limit)

Some of the terminologies are given here –
- **Entities :** Customers or servers
- **Attributes:** Arrival time, time in queue, service time, status
- **Activity:** Waiting in queue, serving
- **State:** Either $x$ ( number of customers in queue) or $y$ (status of the server – 0 or 1)
- **Event:** Arrival event, Departure event

When a departure event occurs, the simulation proceeds in the manner as shown in Figure 4.2. Flow diagram for arrival event is shown in Figure 4.3.
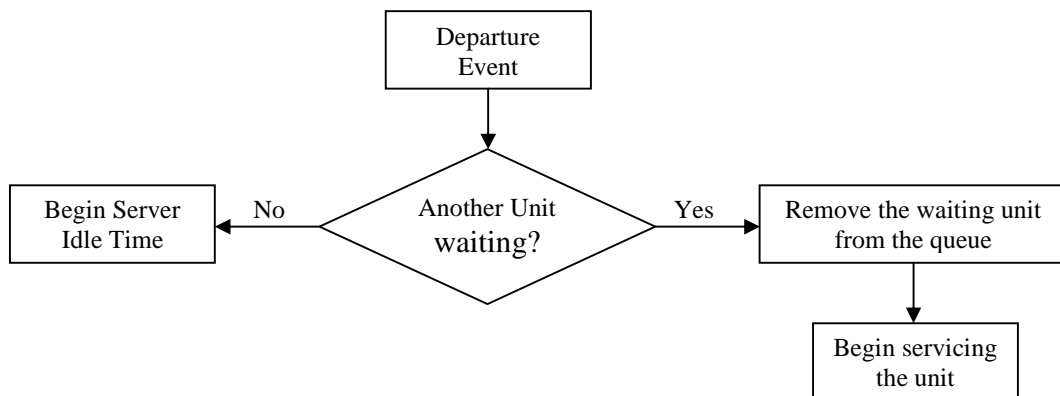
Figure 4.2 Flow diagram for departure event
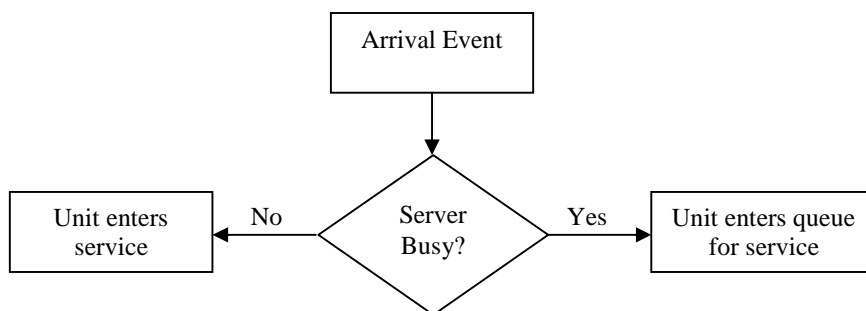
Figure 4.3 Flow diagram for arrival event

### 4.3.2 Simulating a Single-Server Queue
Here, we will discuss single-server queue using an example and compute some performance measures like average waiting time of a customer, idle time of the server etc.

**Example 1:** A small grocery store has one checkout counter. Customers arrive at the checkout counter at random times that range from 1 to 8 minutes apart. We assume that

the inter-arrival times are integer-valued with each of the 8 values having equal probability, following uniform distribution. The service times vary from 1 to 6 minutes (integer) with the probabilities given in the following table:

| Service Time (in Mins) | Probability |
|---|---|
| 1 | 0.10 |
| 2 | 0.20 |
| 3 | 0.30 |
| 4 | 0.25 |
| 5 | 0.10 |
| 6 | 0.05 |

Simulate the system for 10 customers and compute waiting time of customers, total time spent by a customer in the system and idle time of the server. The random numbers for inter-arrival time are: 913, 727, 015, 948, 309, 922, 753, 235, 302. The random numbers for service time are: 84, 10, 74, 53, 17, 79, 91, 67, 89, 38.

**Solution:**
Step (1) It is given that, the customers arrive in the random time 1 – 8 mins apart and each has equal probability. Hence, the probability of arrival time is 1/8 = 0.125. The cumulative probability is computed in Col3 of the Table 4.1. The range of random numbers are listed based on the respective cumulative probability.

Table 4.1 Arrival Time

| Arrival Time in Mins (1) | Probability (2) | Cumulative Probability (3) | Range for Random numbers (4) |
|---|---|---|---|
| 1 | 0.125 | 0.125 | 0 – 125 |
| 2 | 0.125 | 0.250 | 126 – 250 |
| 3 | 0.125 | 0.375 | 251 – 375 |
| 4 | 0.125 | 0.500 | 376 – 500 |
| 5 | 0.125 | 0.625 | 501 – 625 |
| 6 | 0.125 | 0.750 | 626 – 750 |
| 7 | 0.125 | 0.875 | 751 – 875 |
| 8 | 0.125 | 1.000 | 876 - 000 |

Step (2) Service time and their respective probabilities are given. Compute the cumulative probability and also the range of random numbers as given in Table 4.2

Table 4.2 Service Time

| Service Time in Mins (1) | Probability (2) | Cumulative Probability (3) | Range for Random numbers (4) |
|---|---|---|---|
| 1 | 0.10 | 0.10 | 0 – 10 |
| 2 | 0.20 | 0.30 | 11 – 30 |
| 3 | 0.30 | 0.60 | 31 – 60 |
| 4 | 0.25 | 0.85 | 61 – 85 |
| 5 | 0.10 | 0.95 | 86 – 95 |
| 6 | 0.05 | 1.00 | 96 – 00 |

Step (3) Computation of inter-arrival time for each customer has the following procedure:
- (i) The inter-arrival time for first customer is zero
- (ii) Take the first random number (of 3 digits in this example) given and compare it to Col4 of Table 4.1 and identify the appropriate row. The value in that row in Col1 of Table 4.1 will be the inter-arrival time of second customer. For example, the first random number is 913. In the Col4 of Table 4.1, it falls into the last row. The respective value in Col1 is 8. Hence, the inter-arrival time of second customer is 8.
- (iii) Perform above step for all the customers. The result is shown in Col2 of Table 4.3.

Step (4) Service time for each customer is computed in a similar manner. For every customer, a random number (two digits list in this example) is selected and verified with Table 4.2. It is shown in Col4 of Table 4.3.

Table 4.3 Simulation Table

| Customer (1) | Inter-arrival time (2) | Arrival time in clock (3) | Service time (4) | Time in clock when service begins (5) | Waiting time in Queue (6) | Time service ends (7) | Time spent by customer in system (8) | Idle time of Server (9) |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 4 | 0 | 0 | 4 | 4 | - |
| 2 | 8 | 8 | 1 | 8 | 0 | 9 | 1 | 4 |
| 3 | 6 | 14 | 4 | 14 | 0 | 18 | 4 | 5 |
| 4 | 1 | 15 | 3 | 18 | 3 | 21 | 6 | 0 |
| 5 | 8 | 23 | 2 | 23 | 0 | 25 | 2 | 2 |
| 6 | 3 | 26 | 4 | 26 | 0 | 30 | 4 | 1 |
| 7 | 8 | 34 | 5 | 34 | 0 | 39 | 5 | 4 |
| 8 | 7 | 41 | 4 | 41 | 0 | 45 | 4 | 2 |
| 9 | 2 | 43 | 5 | 45 | 2 | 50 | 7 | 0 |
| 10 | 3 | 46 | 3 | 50 | 4 | 53 | 7 | 0 |
| Total | 46 | | 35 | | 9 | | 44 | 18 |

Step(5) The time in the clock when the service begins is treated as 0 for the first customer and for all other (Col 5 in Table 4.3) customers (i.e. from Row2 onwards), it is calculated as–

Max{ sum of Col 4 and Col 5 values in previous row, value in Col 3 of current row}

That is, for second row, it would be –

Max{0+4,8} = 8

Step(6) Waiting time in a queue is :  Col 6 = Col 5 – Col 3
Time service ends is: Col 7 = Col 4 + Col 5
Time spent by customer in system is: Col 8 = Col 7 – Col 3

Idle time of Server (Col 9) for $1^{st}$ customer is not considered. From second customer onwards, it is –

Col 9 = value of Col 5 in current row – value of Col 7 in previous row

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Once the simulation table (Table 4.3) is formulated, we need to compute the performance measures for the waiting line model –

- **Average waiting time =**
    **Total time spent by customers in queue / total number of customers**
    **= 9/10 = 0.9 minutes**
- **The probability that a customer has to wait in the queue =**
    **Number of customers who wait / total number customers**
    **= 3/10 = 0.3**
- **Probability of idle server = total idle time of server / total run time of simulation**
    **= 18/53 = 0.3396   (34% aprox)**
    That is, server is busy for about 66% of the time.
- **The average service time = total service time / total number of customers**
    **= 35/10 = 3.5 minutes**
    This result is out of simulation. The expected service time can be computed by multiplying Col 1 and Col 2 in each row of Table 4.2 and then adding all those values. In this example, it would be –
    1*(0.1) + 2*(0.2) + 3*(0.3) + 4*(0.25) + 5*(0.1) + 6*(0.05) = 3.5 minutes
- **The average inter-arrival time =**
    **sum of all inter-arrival times / number of arrivals – 1**
    **= 46 / 9 = 5.11 minutes**
- **The average waiting time of those who wait =**
    **Total time spent by customers in queue / total number of customers who wait**
    **= 9/3 = 3**
- **The average time spent by customer in a system =**
    **total time spent by customers in the system / total number of customers**
    **= 44/10 = 4.4 minutes**
    This can be also calculated as –
    Average waiting time + Average service time = 0.9 + 3.5  = 4.4 minutes

**Example 2:**   Students arrive at the university library counter wither inter-arrival times distributed as –

| Time between arrival (min) | Probability |
|---|---|
| 5 | 0.2 |
| 6 | 0.7 |
| 7 | 0.1 |

The time for a service (transaction) at the counter is distributed as –

| Service time (min) | Probability |
|---|---|
| 2 | 0.15 |
| 3 | 0.60 |
| 4 | 0.15 |
| 5 | 0.10 |

Simulate the operation for 10 new students. Use the following random numbers for inter-arrival time (95, 76, 35, 21, 46, 87, 18, 65, 58)  and service time (85, 55, 78, 65, 45, 36, 28, 19, 65, 9). Find the average of IAT, waiting time, spent in system time, probability of idle time.

**Solution:** The possible time between arrivals with respective probabilities are given in the question. Hence, the table "Arrival time" would be as below –

| Time between arrival (min) | Probability | Cumulative probability | Random number range |
|---|---|---|---|
| 5 | 0.2 | 0.2 | 00 – 20 |
| 6 | 0.7 | 0.9 | 21 – 90 |
| 7 | 0.1 | 1.0 | 91 – 00 |

The service time and respective probabilities are given. Hence, the table "service time" would be as below –

| Service time (min) | Probability | Cumulative probability | Random number range |
|---|---|---|---|
| 2 | 0.15 | 0.15 | 00- 15 |
| 3 | 0.60 | 0.75 | 16 – 75 |
| 4 | 0.15 | 0.90 | 76 – 90 |
| 5 | 0.10 | 1.00 | 91 - 00 |

Based on the random numbers given for inter-arrival time and service time, the values for Col (2) and Col (4) for the Simulation table (given below) are found. Remaining columns are computed as explained in the previous Example 1.

| Customer

(1) | Inter-arrival time

(2) | Arrival time in clock

(3) | Service time

(4) | Time in clock when service begins
(5) | Waiting time in Queue

(6) | Time service ends

(7) | Time spent by customer in system
(8) | Idle time of Server

(9) |
|---|---|---|---|---|---|---|---|---|
| 1 | - | 0 | 4 | 0 | 0 | 4 | 4 | - |
| 2 | 7 | 7 | 3 | 7 | 0 | 10 | 3 | 3 |
| 3 | 6 | 13 | 4 | 13 | 0 | 17 | 4 | 3 |
| 4 | 6 | 19 | 3 | 19 | 0 | 22 | 3 | 2 |
| 5 | 6 | 25 | 3 | 25 | 0 | 28 | 3 | 3 |
| 6 | 6 | 31 | 3 | 31 | 0 | 34 | 3 | 3 |
| 7 | 6 | 37 | 3 | 37 | 0 | 40 | 3 | 3 |
| 8 | 5 | 42 | 3 | 42 | 0 | 45 | 3 | 2 |
| 9 | 6 | 48 | 3 | 48 | 0 | 51 | 3 | 3 |
| 10 | 6 | 54 | 2 | 54 | 0 | 56 | 2 | 3 |
| Total | 54 | | 31 | | 0 | | 31 | 25 |

**Average inter-arrival time = 54/9 = 6 mins**
**Average waiting time = 0/10 = 0 mins**
**Average time spent in system = 31/10 = 3.1 mins**
**Probability of idle time =  25 /56= 0.4464**

**Example 3:**  The inter-arrival time and service times of customers in a single channel queuing system as follows:

| Customer | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Inter-Arrival Time (mins) | -- | 1 | 2 | 4 | 5 | 7 | 3 | 5 | 6 | 2 |
| Service Time (mins) | 4 | 2 | 3 | 1 | 3 | 4 | 5 | 2 | 2 | 4 |

Calculate (i) Probability of idle server   (ii) Average service time.

**Solution:** The inter-arrival time and service time are given. Hence, we need to just formulate the simulation table.

| Customer (1) | Inter-arrival time (2) | Arrival time in clock (3) | Service time (4) | Time in clock when service begins (5) | Waiting time in Queue (6)= (5)-(3) | Time service ends (7)= (4)+(5) | Time spent by customer in system (8)= (7)-(3) | Idle time of Server (9) |
|---|---|---|---|---|---|---|---|---|
| 1 | - | 0 | 4 | 0 | 0 | 4 | 4 | - |
| 2 | 1 | 1 | 2 | 4 | 3 | 6 | 5 | 0 |
| 3 | 2 | 3 | 3 | 6 | 3 | 9 | 6 | 0 |
| 4 | 4 | 7 | 1 | 9 | 2 | 10 | 3 | 0 |
| 5 | 5 | 12 | 3 | 12 | 0 | 15 | 3 | 2 |
| 6 | 7 | 19 | 4 | 19 | 0 | 23 | 4 | 4 |
| 7 | 3 | 22 | 5 | 23 | 1 | 28 | 6 | 0 |
| 8 | 5 | 27 | 2 | 28 | 1 | 30 | 3 | 0 |
| 9 | 6 | 33 | 2 | 33 | 0 | 35 | 2 | 3 |
| 10 | 2 | 35 | 4 | 35 | 0 | 39 | 4 | 0 |
| Total | 35 | | 30 | | 10 | | | 9 |

(i) Probability of idle server   = 9/10 = 0.9
(ii) Average service time = 30/10 = 3.0 minutes

### 4.3.3  Simulating a Queue with Two Servers
When there are two servers, their service times may be different. And the customers may join any queue (out of two). Hence, the performance of the system will be different from that of singer-server queue. This concepts is discussed using following example –

**Example: The Able-Baker Call Center Problem**
A computer technical support center is staffed by two people – Able and Baker who take calls and try to answer questions and solve computer problems. The time between calls ranges from 1 to 4 minutes, with distribution as shown in the table given below –

| Inter-arrival time (mins) | Probability |
|---|---|
| 1 | 0.25 |
| 2 | 0.40 |
| 3 | 0.20 |
| 4 | 0.15 |

Able is more experienced and can provide service faster than Baker. The distributions of their service times are shown in following tables. When both are idle, Able takes the call. If both are busy, the customer has to wait.

Distribution of Able's Service Time

| Service time (mins) | Probability |
|---|---|
| 2 | 0.30 |
| 3 | 0.28 |
| 4 | 0.25 |
| 5 | 0.17 |

Distribution of Baker's Service Time

| Service time (mins) | Probability |
|---|---|
| 3 | 0.35 |
| 4 | 0.25 |
| 5 | 0.20 |
| 6 | 0.20 |

Simulate the system for 15 customers and compute average inter-arrival time, average service time of Able, average service time of Baker, average waiting time of customers, probability of idle time of Able, probability of idle time of Baker. The random numbers for inter-arrival time are: 26, 98, 90, 26, 42, 74, 80, 68, 22, 48, 34, 45, 24, 34. The random numbers for service time are: 95, 21, 51, 92, 89, 38, 13, 61, 50, 49, 39, 53, 88, 01, 81.

**Solution:** Initially, the cumulative probability and range of random numbers are computed for inter-arrival time and service time of both Able and Baker.

Inter-arrival Time Table

| Inter-arrival time (mins) | Probability | Cumulative Probability | Range of Random Numbers |
|---|---|---|---|
| 1 | 0.25 | 0.25 | 0 – 25 |
| 2 | 0.40 | 0.65 | 26 – 65 |
| 3 | 0.20 | 0.85 | 66 – 85 |
| 4 | 0.15 | 1.00 | 86 – 00 |

Distribution of Able's Service Time

| Service time (mins) | Probability | Cumulative Probability | Range of Random Numbers |
|---|---|---|---|
| 2 | 0.30 | 0.30 | 0 – 30 |
| 3 | 0.28 | 0.58 | 31 – 58 |
| 4 | 0.25 | 0.83 | 59 – 83 |
| 5 | 0.17 | 1.00 | 84 – 00 |

Distribution of Baker's Service Time

| Service time (mins) | Probability | Cumulative Probability | Range of Random Numbers |
|---|---|---|---|
| 3 | 0.35 | 0.35 | 0 – 35 |
| 4 | 0.25 | 0.60 | 36 – 60 |
| 5 | 0.20 | 0.80 | 61 – 80 |
| 6 | 0.20 | 1.00 | 81 – 00 |

Simulation table is prepared as instructed below –
   (i) Inter-arrival time (Col2) is computed using the given random numbers and "Inter-arrival Time Table".
   (ii) Time in Clock (Col3) is cumulative sum of Col2.
   (iii) Col4 is filled with random numbers given for service time.
   (iv) As it is given that, when both Able and Baker are free, Able will serve the customer, Service begins for Able (Col5) is 0. Use the random number (Col4) and compare it with "Distribution of Able's Service Time" table to get the Col6 value of 1st row. Col7= Col5 + Col6
   (v) Observe when next customer arrives (Col3). Allot him to either Able or Baker depending on their availability. Here, 2nd customer is assigned to Baker, as Able was busy. Find Col9 accordingly. Col10 is computed using the random number. Col11= Col9+Col10.
   (vi) Perform the procedure for all customers.
   (vii) Idle time of Able (Col8) = Col5 of next served customer – Col7 of previously served customer.

Similarly, Idle time of Baker(Col12) = Col9 of next served customer – Col11 of previously served customer
(viii)    Time in Queue (Col13) = Either of Col5 or Col9 – Col3

The resulting table is given below.

Simulation Table

| Cust No (1) | Inter-arrival time (2) | Arrival time in Clock (3) | Random Number for Service Time (4) | Able | | | | Baker | | | | Time in Q (13) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Service time begins (5) | Service time (6) | Service time ends (7) | Idle time (8) | Service time begins (9) | Service time (10) | Service time ends (11) | Idle time (12) | |
| 1 | - | 0 | 95 | 0 | 5 | 5 | 0 | | | | | 0 |
| 2 | 2 | 2 | 21 | | | | | 2 | 3 | 5 | 2 | 0 |
| 3 | 4 | 6 | 51 | 6 | 3 | 9 | 1 | | | | | 0 |
| 4 | 4 | 10 | 92 | 10 | 5 | 15 | 1 | | | | | 0 |
| 5 | 2 | 12 | 89 | | | | | 12 | 6 | 18 | 7 | 0 |
| 6 | 2 | 14 | 38 | 15 | 3 | 18 | 0 | | | | | 1 |
| 7 | 3 | 17 | 13 | 18 | 2 | 20 | 0 | | | | | 1 |
| 8 | 3 | 20 | 61 | 20 | 4 | 24 | 0 | | | | | 0 |
| 9 | 3 | 23 | 50 | | | | | 23 | 4 | 27 | 5 | 0 |
| 10 | 1 | 24 | 49 | 24 | 3 | 27 | 0 | | | | | 0 |
| 11 | 2 | 26 | 39 | 27 | 3 | 30 | 0 | | | | | 1 |
| 12 | 2 | 28 | 53 | | | | | 28 | 4 | 32 | 1 | 0 |
| 13 | 2 | 30 | 88 | 30 | 5 | 35 | 0 | | | | | 0 |
| 14 | 1 | 31 | 01 | | | | | 32 | 3 | 35 | 0 | 1 |
| 15 | 2 | 33 | 81 | 35 | 4 | 39 | 0 | | | | | 2 |
| Total | 33 | | | | 37 | | 2 | | 20 | | 15 | 6 |

Performance Measures:
(i)   Average Inter-arrival time = 33/14 = 2.36 mins
(ii)  Average service time of Able =
        Total service time of Able / Number of people Able served = 37/10  = 3.7 mins
(iii) Average service time of Baker =
        Total service time of Baker / Number of people Baker served = 20/5 = 4 mins
(iv) Average waiting time of customers = Total waiting time / Number of customers
                                = 6/15 = 0.4 mins
(v) Average waiting time of customers who wait =
                    total waiting time / number of customers who wait
                    = 6/5 = 1.2 mins
(vi) Probability of Able's idle time =
        Able's idle time / total run time of simulation = 2/39  = 0.05
(vii)    Probability of Baker's idle time
                    = Baker's idle time / total run time of simulation = 15/39 =0.385
(viii)    Probability that a customer has to wait in a queue =
            Number of customers who wait / total number of customers  = 5/15 = 0.33

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## 4.4　SIMULATION EXAMPLE: INVENTORY SYSTEM

An inventory system has number of potential sources of revenue and costs like revenue from sales, cost price of the items, shipping cost, damage cost, profit by selling scrap etc. Inventory systems have some controllable variables (or policy variables) like

- The maximum inventory level, M
- The review period, N
- The order quantity, Q
- Lead time (delay) for replacement, L

In this section, we will discuss two types of inventory problems through examples.

**Example: The News Dealer's Problem**

A news dealer buys papers for 33 cents each and sells them for 50 cents each. Newspapers not sold at the end of the day are sold as scrap for 5 cents each. Newspapers can be purchased in bundles of 10. The order quantity Q is the only policy decision. Q is fixed since ending inventory is always zero due to scrapping leftover papers. There are 3 types of newsdays: "good", "fair", "poor", but the news dealer cannot predict which type will occur on any given day. Following table gives the distribution of newspapers demanded by the type of day.

| Demand | Demand probabilities | | |
|---|---|---|---|
| | Good | Fair | Poor |
| 40 | 0.03 | 0.10 | 0.44 |
| 50 | 0.05 | 0.18 | 0.22 |
| 60 | 0.15 | 0.40 | 0.16 |
| 70 | 0.20 | 0.20 | 0.12 |
| 80 | 0.35 | 0.08 | 0.06 |
| 90 | 0.15 | 0.04 | 0.00 |
| 100 | 0.07 | 0.00 | 0.00 |

The distribution for type of newsday is given as below –

| Type | Probability |
|---|---|
| Good | 0.35 |
| Fair | 0.45 |
| Poor | 0.20 |

Simulate for the purchase of 70 news papers per day for 20 days and compute the profit of news dealer.　Random numbers for Type of News day: 94, 77, 49, 45, 43, 32, 49, 00, 16, 24, 31, 14, 41, 61, 85, 08, 15, 97, 52, 78
Random numbers for Demand: 80, 20, 15, 88, 98, 65, 86, 73, 24, 60, 60, 29, 18, 90, 93, 73, 21, 45, 76, 96

**Solution:** Compute the cumulative probabilities for Demand Table and Type of Newsday Table as below –

Demand Table

| Demand | Demand probabilities | | | Cumulative probabilities | | | Range of Random Numbers | | |
|---|---|---|---|---|---|---|---|---|---|
| | Good | Fair | Poor | Good | Fair | Poor | Good | Fair | Poor |
| 40 | 0.03 | 0.10 | 0.44 | 0.03 | 0.10 | 0.44 | 0 – 3 | 0 – 10 | 0 – 44 |
| 50 | 0.05 | 0.18 | 0.22 | 0.08 | 0.28 | 0.66 | 4 – 8 | 11 – 28 | 45 – 66 |
| 60 | 0.15 | 0.40 | 0.16 | 0.23 | 0.68 | 0.82 | 9 – 23 | 29 – 68 | 67 – 82 |
| 70 | 0.20 | 0.20 | 0.12 | 0.43 | 0.88 | 0.94 | 24 – 43 | 69 – 88 | 83 – 94 |
| 80 | 0.35 | 0.08 | 0.06 | 0.78 | 0.96 | 1.00 | 44 – 78 | 89 – 96 | 95 – 00 |
| 90 | 0.15 | 0.04 | 0.00 | 0.93 | 1.00 | 1.00 | 79 – 93 | 97 - 00 | |
| 100 | 0.07 | 0.00 | 0.00 | 1.00 | 1.00 | 1.00 | 94 - 00 | | |

Distribution of Type of Newsday

| Type | Probability | Cumulative Probability | Range of Random Numbers |
|------|-------------|------------------------|-------------------------|
| Good | 0.35 | 0.35 | 01 – 35 |
| Fair | 0.45 | 0.80 | 36 – 80 |
| Poor | 0.20 | 1.00 | 81 – 00 |

Profit is calculated using the formula –
**Profit = Revenue from sales – Cost of News papers – lost profit from excess demand
+ Salvage from sale of scrap papers**

The News dealer buys 70 news paper every day for price 33 cents. Hence,
Cost of News Paper = 70 * 0.33 = 23.1 for all days.

The columns of the simulation table will be calculated as follows –
  (i)  Col2 and Col4 are filled with the random numbers given in the question for the
         respective purpose.
  (ii) Random numbers in Col2 are compared with "Distribution of Type of Newsday" table
         and appropriate type is filled in Col3.
  (iii) Random numbers in Col4 are compared with Col3 and "Demand Table" to get the
         appropriate demand. This is filled in Col5.
  (iv)  If Col5 <=70, then
            Col6 = Col5 * 0.5
        Else
            Col6 = 70 *0.5 = 35.0
  (v) If there is an excess demand (than 70 papers), then the news dealers loss would be
        the difference of selling price and cost price = 50 – 33 = 17 cents.  Hence,
            If Col5 > 70 then
                Col7 = (Col5 – 70) * 0.17
            Else
            Col7=0
  (vi) If there is a less demand (than 70 papers), then he will sell the remaining papers for
        5 cents. Hence,
            If Col5<70 then
                Col8 = (70 - Col5) * 0.05
            Else
                Col8 = 0
  (vii)    Col 9 = Col 6 – 23.1 – Col 7 + Col 8

From the simulation table given below, it is found that **Profit = 174.9 Dollars.**

| Day (1) | Random Numbers for Type of Day (2) | Type of Day (3) | Random Numbers for Demand (4) | Demand (5) | Revenue from Sales (6) | Loss from excess demand (7) | Sale of scrap (8) | Profit (9) |
|---|---|---|---|---|---|---|---|---|
| 1 | 94 | Poor | 80 | 60 | 30 | 0 | 0.5 | 7.4 |
| 2 | 77 | Fair | 20 | 50 | 25 | 0 | 1 | 2.9 |
| 3 | 49 | Fair | 15 | 50 | 25 | 0 | 1 | 2.9 |
| 4 | 45 | Fair | 88 | 70 | 35 | 0 | 0 | 11.9 |
| 5 | 43 | Fair | 98 | 90 | 35 | 3.4 | 0 | 8.5 |
| 6 | 32 | Good | 65 | 80 | 35 | 1.7 | 0 | 10.2 |
| 7 | 49 | Fair | 86 | 70 | 35 | 0 | 0 | 11.9 |
| 8 | 00 | Poor | 73 | 60 | 30 | 0 | 0.5 | 7.4 |
| 9 | 16 | Good | 24 | 70 | 35 | 0 | 0 | 11.9 |
| 10 | 24 | Good | 60 | 80 | 35 | 1.7 | 0 | 10.2 |
| 11 | 31 | Good | 60 | 80 | 35 | 1.7 | 0 | 10.2 |
| 12 | 14 | Good | 29 | 70 | 35 | 0 | 0 | 11.9 |
| 13 | 41 | Fair | 18 | 50 | 25 | 0 | 1 | 2.9 |
| 14 | 61 | Fair | 90 | 80 | 35 | 1.7 | 0 | 10.2 |
| 15 | 85 | Poor | 93 | 70 | 35 | 0 | 0 | 11.9 |
| 16 | 08 | Good | 73 | 80 | 35 | 1.7 | 0 | 10.2 |
| 17 | 15 | Good | 21 | 60 | 30 | 0 | 0.5 | 7.4 |
| 18 | 97 | Poor | 45 | 50 | 25 | 0 | 1 | 2.9 |
| 19 | 52 | Fair | 76 | 70 | 35 | 0 | 0 | 11.9 |
| 20 | 78 | Fair | 96 | 80 | 35 | 1.7 | 0 | 10.2 |
| Total | | | | | 645 | 13.6 | 5.5 | 174.9 |

**Simulating an (M, N) Inventory Policy:**
Here, there will be a fixed review period (N) and a maximum inventory level (M). That is, the review of pending stock is done in a fixed intervals and the purchase is made only up to certain level. Demand and lead time are random.

**Example: A Refrigerator Inventory Problem**
A company selling refrigerators maintains inventory by conducting a review for every 5 days and it keeps the order-up-to-level as 11 refrigerators. The distribution of number of units demanded per day are given as below –

| Demand | Probability |
|---|---|
| 0 | 0.10 |
| 1 | 0.25 |
| 2 | 0.35 |
| 3 | 0.21 |
| 4 | 0.09 |

The distribution for lead time is given as –

| Lead Time (Days) | Probability |
|---|---|
| 1 | 0.6 |
| 2 | 0.3 |
| 3 | 0.1 |

Starting inventory is 3 and an order has been placed for 8 refrigerators, which is supposed to be delivered in next 2 days. By simulation, estimate average ending units in inventory and number of days when shortage occurs. Random Numbers for Demand: 24, 35, 65, 81, 54, 03, 87, 27, 73, 70, 47, 45, 48, 17, 09, 42, 87, 26, 36, 40, 07, 63, 19, 88, 94.  Random numbers for lead time: 5, 0, 3, 4,8

**Solution:** Compute the cumulative probability and range of random numbers for demand and lead time.

Distribution for number of units demanded

| Demand | Probability | Cumulative Probability | Range of Random Numbers |
|--------|-------------|------------------------|-------------------------|
| 0 | 0.10 | 0.10 | 01 – 10 |
| 1 | 0.25 | 0.35 | 11 – 35 |
| 2 | 0.35 | 0.70 | 36 – 70 |
| 3 | 0.21 | 0.91 | 71 – 91 |
| 4 | 0.09 | 1.00 | 92 - 00 |

Distribution for lead time

| Lead Time (Days) | Probability | Cumulative Probability | Range of Random Numbers |
|------------------|-------------|------------------------|-------------------------|
| 1 | 0.6 | 0.6 | 1 – 6 |
| 2 | 0.3 | 0.9 | 7 – 9 |
| 3 | 0.1 | 1.0 | 0 |

The simulation is done for 25 days consisting of 5 cycles. It is given in the question that, at the end of previous cycle, the ending inventory is 3 and an order is placed for 8 refrigerators which are supposed to be delivered in next 2 days. This information is mentioned as cycle 0 in the simulation table given below.  Note that, the new order always will be added to the beginning inventory of the day when the stock arrives.

To prepare simulation table for 5 cycles, following steps are done –
   (i)  Start with cycle 1, first day. The beginning inventory would be 3 (as ending inventory of previous cycle is 3) – mentioned in Col3. Take a random number (Col4) and compare it with the table "Distribution of number of units demanded". As the demand happens to be 1, put it in Col5.
         Ending inventory (Col6) = Co3 – Col5  = 3 – 1 = 2
      Number of days left for new stock delivery (Col10) would be now 1.
   (ii) For day2, starting inventory (Col3) is ending inventory of previous day, i.e.  2. Take next random number and find out the demand (Col5).  Compute Col6 as before. Number of days left for new stock is now 0. That is, for next day morning, it will be available.
   (iii) For day3, Beginning inventory (Col3) would be ending inventory of previous day + order received. That is, 1+8 =9. Now, compute Col5 and Col6 as before till 5$^{th}$ day.
   (iv) At the end of Day5, ending inventory is 2. Hence, new order is placed for 11-2 = 9. Note that, the order-up-to-level is 11, as given in the question. Hence 9 is placed in Col8. Now, the random number 5 for lead time is taken and placed in Col9. It is compared with the table "Distribution for lead time" to get the lead time. It is found to be 1 and is placed in Col10.
   (v) Procedure is continued for all 25 days of 5 cycles. One can observe from the simulation table that, there will be a shortage of stock on Day2 and Day3 of Cycle3, because of delay in supply. Still, no change is made in the system and order is always placed at the end of 5$^{th}$ day. The shortage is fulfilled when the new order arrives as shown in Day4 of Col3. Here, starting inventory (including new stock) is 9 and the demand is 1. Still, the ending inventory would be 4 because, pending request of 4 refrigerators is delivered now.

| Cycle (1) | Day (2) | Beginning Inventory (3) | Random Number for Demand (4) | Demand (5) | Ending Inventory (6) | Shortage Quantity (7) | Order Quantity (8) | Random number for lead time (9) | Days until order arrives (10) |
|---|---|---|---|---|---|---|---|---|---|
| *0* | *5* | | | | *3* | *-* | *8* | | *2* |
| 1 | 1 | 3 | 24 | 1 | 2 | - | - | - | 1 |
| | 2 | 2 | 35 | 1 | 1 | | | | 0 |
| | 3 | 9 | 65 | 2 | 7 | | | | |
| | 4 | 7 | 81 | 3 | 4 | | | | |
| | 5 | 4 | 54 | 2 | 2 | | 9 | 5 | 1 |
| 2 | 1 | 2 | 03 | 0 | 2 | | | | 0 |
| | 2 | 11 | 87 | 3 | 8 | | | | |
| | 3 | 8 | 27 | 1 | 7 | | | | |
| | 4 | 7 | 73 | 3 | 4 | | | | |
| | 5 | 4 | 70 | 2 | 2 | | 9 | 0 | 3 |
| 3 | 1 | 2 | 47 | 2 | 0 | | | | 2 |
| | 2 | 0 | 45 | 2 | 0 | 2 | | | 1 |
| | 3 | 0 | 48 | 2 | 0 | 4 | | | 0 |
| | 4 | 9 | 17 | 1 | 4 | | | | |
| | 5 | 4 | 09 | 0 | 4 | | 7 | 3 | 1 |
| 4 | 1 | 4 | 42 | 2 | 2 | | | | 0 |
| | 2 | 9 | 87 | 3 | 6 | | | | |
| | 3 | 6 | 26 | 1 | 5 | | | | |
| | 4 | 5 | 36 | 2 | 3 | | | | |
| | 5 | 3 | 40 | 2 | 1 | | 10 | 4 | 1 |
| 5 | 1 | 1 | 07 | 0 | 1 | | | | 0 |
| | 2 | 11 | 63 | 2 | 9 | | | | |
| | 3 | 9 | 19 | 1 | 8 | | | | |
| | 4 | 8 | 88 | 3 | 5 | | | | |
| | 5 | 5 | 94 | 4 | 1 | | 10 | 8 | 2 |
| **Total** | | | | | **88** | | | | |

Average ending inventory = 88/25 =3.52
Number of days when there was a shortage = 2