1.) The probability mass function of variable x is given the following table.

X	0	1	2	3	4	5	6
P(x)	k	3k	5k	7k	9k	11k	13k

a) For what value of k does this represent a valid probability distribution? Also find $P(X<4)$, $P(X<4)$, $P(X>=5)$ & $P(3<X<=6)$

b) Determine the minimum value of k so that $P(X \leq 2) >= 0.3$

Ans: a) The total probability is always 1

Hence,

$$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$\Rightarrow k(1+3+5+7+9+11+13) = 1$$

$$\Rightarrow k = \frac{1}{49}$$

$$P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49}$$

$$= \frac{1}{49}\{1+3+5+7\}$$

$$P(X<4) = \underline{\underline{\frac{16}{49}}}$$

$$P(X>=5) = P(X=5) + P(X=6)$$
$$= \frac{11}{49} + \frac{13}{49}$$
$$= \frac{1}{49}(11+13)$$
$$P(X>=5) = \frac{24}{49}$$

$$P(3 < X \leq 6) = P(X=4) + P(X=5) + P(X=6)$$
$$= \frac{9}{49} + \frac{11}{49} + \frac{13}{49}$$
$$= \frac{1}{49}(9+11+13)$$
$$P(3 < X \leq 6) = \frac{33}{49}$$

b) $P(X \leq 2) = P(0) + P(1) + P(2)$
$$= k + 3k + 5k$$
$$= 9k.$$

$\therefore P(X \leq 2) \geq 0.3$

$\Rightarrow 9k \geq 0.3$

$\Rightarrow k \geq \frac{0.3}{9}$

$\Rightarrow k \geq \frac{1}{30}$

Thus, $\frac{1}{30}$ is the minimum value of k so that $P(X \leq 2) \geq 0.3$.

2) A fair coin is tossed 3 times. Let X denote the number of heads showing up. Find the distribution of X. Also find its Mean, variance and standard deviation.

Ans :- When a coin is tossed 3 times the possible outputs are :

HHH, HHT, HTH, THH,

THT, TTH, HTT, TTT

Total = 8

probability of any of these possibilities $= \frac{1}{8}$

Let $X \to$ no of head showing up

$\Rightarrow R_X = \{0, 1, 2, 3\}$

0 → {TTT}, 1 → {HTT, THT, TTH}, 2 → {THH, HHT, HTH}, 3 → {HHH}

Distribution is :

X	0	1	2	3
p(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean $= E(X) = \sum_{\forall i} x_i \, p(x_i)$

$= \{0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8}\}$

$= \frac{1}{8} \{0 + 3 + 6 + 3\}$

$= \frac{12}{8} = \underline{1.5}$

Variance

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \sum_{\forall i} x_i^2 \; p(x_i) - [E(x)]^2$$

$$= \left\{ 0^2 \frac{1}{8} + 1^2 \frac{3}{8} + 2^2 \frac{3}{8} + 3^2 \frac{1}{8} \right\} - (1.5)^2$$

$$= \frac{1}{8} \{0 + 3 + 12 + 9\} - 2.25$$

$$= \frac{24}{8} - 2.25$$

$$\boxed{V(x) = 0.75}$$

Standard deviation

$$= \sqrt{V(x)}$$

$$= \sqrt{0.75}$$

$$= \underline{0.8660}$$

3) Random variable X has the continuous function :

$$f(x) = kx^2, \quad 0 \le x \le 3$$

Evaluate k and find $P(X <= 1)$, $P(1 <= X <= 2)$, $P(X <= 2)$, $P(x > 1)$ $P(X > 2)$. Compute Mean & variance.

Ans :- $f(x) = kx^2, \quad 0 \le x \le 3$

$$\Rightarrow \int_0^3 kx^2 \, dx = 1$$

$$\Rightarrow k \left. \frac{x^3}{3} \right|_0^3 = 1$$

$$\Rightarrow k \left\{ \frac{3^3}{3} - \frac{0^3}{3} \right\} = 1$$

$$\Rightarrow k = \frac{1}{9}$$

Hence, $f(x) = \frac{x^2}{9}, \quad 0 \le x \le 3$

$$P(x \le 1) = \int_0^1 f(x) \, dx$$

$$= \int_0^1 \frac{x^2}{9} \, dx$$

$$= \frac{1}{9} \left\{ \frac{x^3}{3} \right\} \Big|_0^1$$

$$= \frac{1}{9} \left\{ \frac{1^3}{3} - \frac{0^3}{3} \right\}$$

$$= \frac{1}{27}$$

$$P(1 \le x \le 2) = \int_1^2 f(x)\,dx$$

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$$= \int_1^2 \frac{x^2}{9}\,dx$$

$$= \frac{1}{9}\left\{\frac{x^3}{3}\right\}\Big|_1^2$$

$$= \frac{1}{9}\left\{\frac{8}{3} - \frac{1}{3}\right\} = \frac{7}{27}$$

$$P(x \le 2) = \int_0^2 f(x)\,dx$$

$$= \int_0^2 \frac{x^2}{9}\,dx$$

$$= \frac{1}{9}\left\{\frac{x^3}{3}\right\}\Big|_0^2$$

$$= \frac{1}{9}\left\{\frac{2^3}{3} - \frac{0^3}{3}\right\}$$

$$= \frac{1}{9}\left\{\frac{8}{3} - 0\right\}$$

$$= \frac{8}{27}$$

$$P(x>1) = 1 - P(x \le 1)$$

$$= 1 - \frac{1}{27} = \frac{26}{27}$$

$$P(x>2) = 1 - P(x \le 2)$$

$$= 1 - \frac{8}{27} = \frac{19}{27}$$

<u>Mean</u> $= E(x) = \int_0^3 x\, f(x)\, dx$

$$= \int_0^3 x \cdot \frac{x^2}{9}\, dx$$

$$= \frac{1}{9}\left\{\frac{x^4}{4}\right\}\Big|_0^3$$

$$= \frac{1}{9}\left\{\frac{3^4}{4} - \frac{0^4}{4}\right\}$$

$$= \frac{9}{4} = 2.25.$$

<u>Variance</u> =

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \int_0^3 x^2 f(x)\, dx - \left[\int x\, f(x)\, dx\right]^2$$

$$= \int_0^3 x^2 \cdot \frac{x^2}{9}\, dx - (2.25)^2$$

$$= \frac{1}{9}\left\{\frac{x^5}{5}\right\}\Big|_0^3 - (2.25)^2$$

$$= \frac{1}{9}\left\{\frac{243}{5}\right\} - 5.0625 = 0.3375$$

4) Given that 2% of the fuses manufactured by a firm are defective, find, by using Poisson distribution, the probability that a box containing 200 fuses has

i) No defective fuse ii) 3 or more defective

iii) at least 1 defective.

<u>Ans</u>:- Given that 2% are defective

$\Rightarrow p = 0.02$

Also, $n = 200$

In poisson distribution,

$$\alpha = n \cdot p$$

$$\alpha = 200 \times 0.02$$

$$\boxed{\alpha = 4}$$

(i) No defective fuse

$\Rightarrow x = 0$

$$P(X=0) = \frac{e^{-\alpha}\alpha^{x}}{x!}$$

$$= \frac{e^{-4}4^{0}}{0!}$$

$$= e^{-4}$$

$$= \underline{0.0183}$$

(ii) 3 or more defective fuses

$P(x \geq 3) = 1 - P(x < 3)$

$= 1 - \{P(x=0) + P(x=1) + P(x=2)]$

$= 1 - \{e^{-4} + e^{-4} \cdot 4 + e^{-4} \cdot 8\}$

$= 1 - 0.2379$

$= \underline{\underline{0.7621}}$

(iii) atleast 1 defective

$P(x \geq 1) = 1 - P(x < 1)$

$= 1 - P(x=0) = 1 - \{e^{-4}\}$

$= 1 - 0.0183$

$= \underline{\underline{0.9817}}$

5) The length of a telephone conversation has an exponential distribution with a mean of 3 minutes. Find a probability that a call i) Ends in less than 3 mins
ii) Takes between 3 and 5 mins.

Ans :- pdf is

$f(x) = \lambda e^{\lambda x}$

Given that -

Mean = 3

$\Rightarrow \frac{1}{\lambda} = 3$

$\Rightarrow \lambda = \frac{1}{3}$

(i) End in less than 3 mins

$P(X \leq 3) = F(3)$

$= 1 - e^{-\lambda x}$

$= 1 - e^{-\frac{1}{3}3}$

$= 1 - e^{-1}$

$= \underline{0.6321}$

(ii) Between 3 and 5 mins

$P(3 \leq X \leq 5) = F(5) - F(3)$

$= \left\{1 - e^{-\frac{1}{3}\cdot 5}\right\} - \left\{1 - e^{-\frac{1}{3}\cdot 3}\right\}$

$= e^{-1} - e^{-5/3}$

$= 0.3678 - 0.1888$

$= \underline{0.179}$

6) A production process manufacture alternators for outboard engines used in recreational boating. On the average, 1% of the alternators will not perform upto the required standards when tested at the engine assembly plant. When a large shipment of alternators is received at the plant, 100 are tested, and if more than two are nonconforming, the shipment is returned to the alternator manufacturer. What is the probability of returning a shipment?

<u>Ans</u>:- Given that -

$$p = 0.01$$

$$n = 100$$

$$\Rightarrow q = 0.99$$

Compute $P(X > 2)$

$$P(X > 2) = 1 - P(X \le 2)$$

$$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$$

$$= 1 - \{ {}^{100}C_0 (0.01)^0 (0.99)^{100-0} + {}^{100}C_1 (0.01)^1 (0.99)^{100-1} + {}^{100}C_2 (0.01)^2 (0.99)^{100-2} \}$$

$$= 1 - \{0.3660 + (100)(0.01)(0.3697) + (4950)(0.0001)(0.3734)\}$$

$$= 1 - 0.9205 = \underline{0.0794}$$

7) Lane Briantwain is quite a popular student. Lane recieves, on the average, four phone calls a night (Poisson-distribution). What is the probability that, tomorrow night, the number of calls recieved will exceed the average by more than standard deviation?

<u>Ans</u>: Given that,

$\alpha = 4 = E(x) = V(x)$

$\Rightarrow SD = \sqrt{V(x)}$

$SD = 2$

$P(X > 6) = ?$

$P(X > 6) = 1 - P(X \leq 6)$

$= 1 - \{P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6)\}$

$= 1 - e^{-4}\left\{1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!}\right\}$

$= 1 - e^{-4}\{1 + 4 + 8 + 10.66 + 10.66 + 8.533 + 5.688\}$

$= 1 - 0.0183\{48.538\}$

$= 1 - 0.888$

$= \underline{0.112}$