DIVIDE AND CONQUER

- We know that some of the problems can be straight-away solved by Brute-Force technique.
- But, in many cases, Brute-force fails. That is, some of the problems can not be solved using Brute force method.
- Here we will study one more algorithm design technique, Divide-and-Conquer.
- The divide and conquer strategy suggests to divide the given problem of size *n* into *k* distinct subproblems (1<k<=n).
- These subproblems must be solved and then a method must be found to combine subsolutions to get the solution of whole problem.
- If the subproblem size is relatively large, then, divide and conquer strategy may be re-applied on that.

- Usually, in divide and conquer strategy, we write a control abstraction that mirrors the way an algorithm will look.
- *Control Abstraction* is a procedure whose flow of control is clear but whose primary operations are specified by other procedures whose precise meanings are left undefined.
- Consider the following algorithm *DAndC*, which is invoked for a problem *P* to be solved.
- *Small(P)* is a Boolean-valued function that determines whether the input size is small enough that the answer can be computed without spitting.

```
Algorithm DAndC(P)
{
    if Small(P) then
      return S(P);
    else
    {
      divide P into smaller instances P1, P2, …, Pk,    k>=1;
      Apply DAndC to each of these subproblems;
      return Combine(DAndC(P1), DAndC(P2), …, DAndC(Pk));
    }
}
```

- If the size of *P* is *n* and the sizes of *k* subproblems are *n1, n2,…,nk*, then the computing time of DAndC is described by the recurrence relation:

$$T(n) = \begin{cases} g(n), & \text{if n is small} \\ T(n1) + T(n2) + \ldots + T(nk) + f(n), & \text{otherwise} \end{cases}$$

- The complexity of many divide and conquer algorithms is given by the recurrences of the form –

$$T(n) = \begin{cases} T(1), & \text{if } n=1 \\ a.\ T(n/b) + f(n), & \text{if } n>1 \end{cases}$$

Merge Sort:

- Merge sort is a best example of DAC technique
- This sorting technique divides a given array A[0…n-1] by dividing it into two parts, viz. A[0…n/2-1] and A[n/2…n-1], Then each of the problems are sorted recursively and finally, they are merged.
- Merging of two sorted arrays is done as follows –
  - Compare first elements of both arrays. Put the smaller element into the resulting array.
  - Now compare remaining element with the 2nd element of other array. Store the smaller into resulting array.
  - Continue the above procedure till one of the arrays get exhausted.
  - Copy all the elements of other array into resulting array.

```
ALGORITHM MergeSort(low, high)
//a[low:high] is an array to be sorted
//Small(P) is true if there is only one element to sort.
{
   if (low<high) then              //if there are more than one element
   {
        //Divide P into subproblems
        mid=(low+high)/2;
        MergeSort(low, mid);              //solve subproblem
        MergeSort(mid+1, high);           //solve subproblem
        Merge(low, mid, high)             //combine the solutions
   }
}
```

```
ALGORITHM Merge(low, mid, high)
//a[low : high] is an array, and two sorted subsets are a[low : mid] and
// a[mid+1 : high]. The goal is to combine these two sets into a single
// set. b[ ] is an auxiliary array.
{
    h:=low, i:=low, j:=mid+1;
    while(h<=mid) and (j<=high) do
    {
        if(a[h]<=a[j]) then
        {
            b[i] :=a[h];
            h :=h+1;
        }
        else
        {
            b[i]:=a[j];
            j := j+1;
        }
    }
    if (h>mid) then
        for k := j to high do
        {
            b[i] :=a[k];
            i :=i+1;
        }
    else
        for k := h to mid do
        {
            b[i] :=a[k];
            i :=i+1;
        }
    for k:= low to high do
        a[k] :=b[k];
}
```

Quick Sort

- Here, the given array is divided into two sub-arrays such that
  - the elements at the left-side of some key element are less than the key element and
  - the elements at the right-side of the key element are greater than the key element.
- The dividing procedure is done with the help of two index variables and one key element as explained below –

1. Usually the first element of the array is treated as *key*. The position of the second element is taken as the first index variable *left* and the position of the last element will be the index variable *right.*
2. Now the index variable *left* is incremented by one till the value stored at the position *left* is greater than the *key*.
3. Similarly *right* is decremented by one till the value stored at the position *right* is smaller than the *key*.

4. Now, these two elements are interchanged. Again from the current position, *left* and *right* are incremented and decremented respectively and exchanges are made appropriately, if required.

5. This process is continued till the index variables either meet or crossover. Now, exchange *key* with the element at the position *right*.

6. Now, the whole array is divided into two parts such that one part is containing the elements less than the *key* element and the other part is containing the elements greater than the *key*. And, the position of *key* is fixed now.

7. The above procedure (from step i to step vi) is applied on both the sub-arrays. After some iteration we will end-up with sub-arrays containing single element. By that time, the array will be stored.

```
ALGORITHM QuickSort (p, q)
//Sorts the elements a[p], …, a[q] which is inside the array a[1: n].
{
   if(p<q) then
   {
        j :=Partition(a, p, q+1);
        QuickSort(p, j-1);
        QuickSort(j+1, q);
   }
}
```

```
ALGORITHM Partition(a, m, p)
{
    v :=a[m];
    i:=m;
    j:=p;

    repeat
    {
        repeat
            i :=i+1;
        until (a[i]>=v);
        repeat
            j :=j-1;
        until (a[ j]<=v);

    if (i<j) then
            Interchange(a, i, j);
    }until (i>=j);
    a[m] := a[j];
    a[j] :=v;
    return j;
}
```

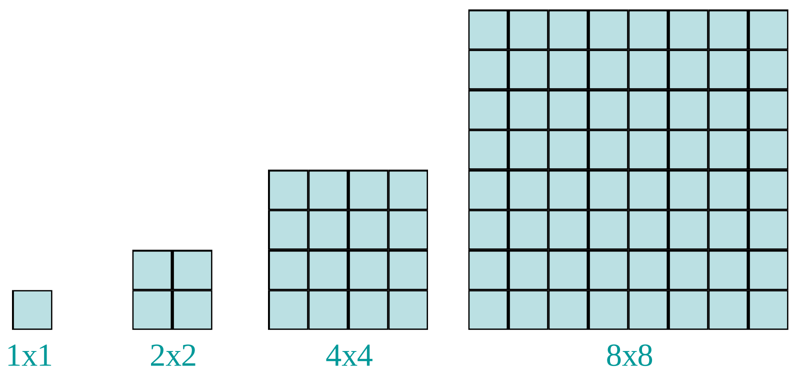
```
Algorithm Interchange(a, i, j)
{
        p:=a[i];
        a[i] := a[ j];
        a[ j] :=p;
}
```

BINARY SEARCH

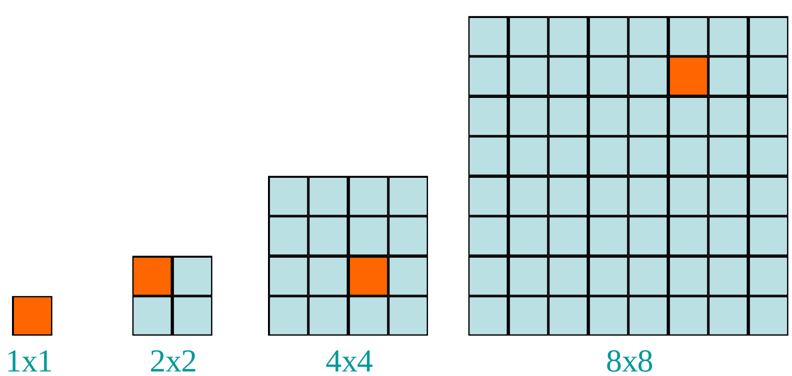
```
ALGORITHM BinSrch(a, i, l, x)
//Given an array a[i:l] of elements in nondecreasing order,
// 1<=i<=l, determine whether x is present. If so, return
//such that x=a[j], else return 0.

{
    if(l==i) then    //If Small(P)
    {
        if  (x==a[i]) then return i;
        else return 0;
    }
    else
    {
        mid:= (i+l)/2;
        if(x==a[mid] then return mid;
        else if (x<a[mid]) then
                return BinSrch(a, i, mid-1, x);
        else    return BinSrch(a, mid+1, l x);
    }
}
```

Defective Chess Board Problem

- A chessboard is an n x n grid, where $n = 2^k$



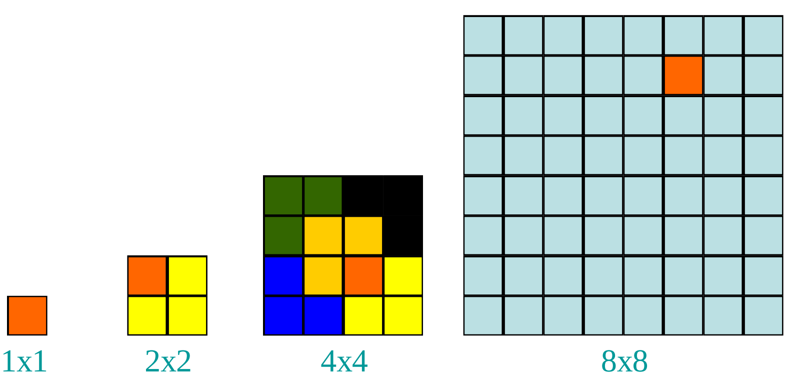
- A defective chessboard is a chessboard that has one unavailable (defective) position.

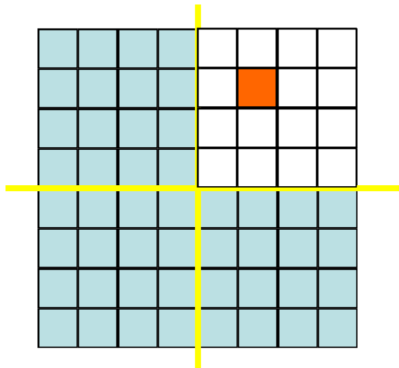


- The problem is to ***tile (cover)*** all non-defective cells using a ***triomino.***
- A triomino is an L shaped object that can cover three squares of a chessboard.
- A triomino has four orientations:

Tiling A Defective Chessboard

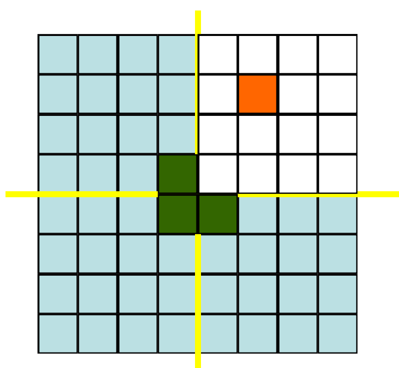
Place $(n^2 - 1)/3$ triominoes on an $n \times n$ defective chessboard so that all $n^2 - 1$ nondefective positions are covered.





Divide into four smaller chessboards. 4 x 4

One of these is a defective 4 x 4 chessboard.



Make the other three 4 x 4 chessboards defective by placing a triomino at their common corner.

Recursively tile the four defective 4 x 4 chessboards.

In general, a 2^k x 2^k defective chessboard can be divided as –

2^{k-1} x 2^{k-1}	2^{k-1} x 2^{k-1}
2^{k-1} x 2^{k-1}	2^{k-1} x 2^{k-1}

Analysis

- Let n = 2^k.
- Let t(k) be the time taken to tile a 2^k x 2^k defective chessboard. Then,

$$t(k) = \begin{cases} 0, & \text{if n} = 0 \\ 4t(k-1) + c, & \text{otherwise} \end{cases}$$

- Here, *c* is constant representing time spent on finding the appropriate position for a triomino and to rotate the triomino for a required shape.

$$
\begin{aligned}
t(k) &= 4t(k-1) + c \\
&= 4[4t(k-2) + c] + c \\
&= 4^2\, t(k-2) + 4c + c \\
&= 4^2[4t(k-3) + c] + 4c + c \\
&= 4^3\, t(k-3) + 4^2c + 4c + c \\
&= \ldots \\
&= 4^k\, t(0) + 4^{k-1}c + 4^{k-2}c + \ldots + 4^2c + 4c + c \\
&= 4^{k-1}c + 4^{k-2}c + \ldots + 4^2c + 4c + c \\
&= \Theta(4^k) \\
&= \Theta(\text{number of triominoes placed})
\end{aligned}
$$